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COMPUTING FUZZY SUBGROUPS OF SOME SPECIAL CYCLIC GROUPS

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Abstract. In this paper, we discuss the number of distinct fuzzy subgroups of the group $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^m} \times \mathbb{Z}_r$, $m = 1, 2, 3$ where p, q, r are distinct primes for any $n \in \mathbb{Z}^+$ using the criss-cut method that was proposed by Murali and Makamba in their study of distinct fuzzy subgroups. The criss-cut method first establishes all the maximal chains of the subgroups of a group G and then counts the distinct fuzzy subgroups contributed by each chain. In this paper, all the formulae for calculating the number of these distinct fuzzy subgroups are given in polynomial form.

1. Introduction

The group $G = \mathbb{Z}_{p^n} \times \mathbb{Z}_{q^m} \times \mathbb{Z}_r$ where p, q, r are distinct primes and $m, n \in \mathbb{Z}^+$ is cyclic. In this paper, we will sometimes use $p^n q^m r$ to denote the group $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^m} \times \mathbb{Z}_r$. We believe that the study of fuzzy groups, including their classification, is very important and interesting because of the importance of fuzzy logic in general. As Murali and Makamba in [5] puts it: one of the most interesting problems in fuzzy group theory is to classify fuzzy subgroups up to some unique invariants of the underlying group. In our classification and counting, we use the equivalence relation defined in [1]. The concept of fuzzy sets was introduced by Zadeh [12] in 1965 and Rosenfeld [9] followed by introducing the concepts of fuzzy subgroupoids and fuzzy subgroups. Fuzzy subgroups have recently been studied by [1,3,4,7] among others, thus extending the work done by the earlier authors like Das in [11] and Sherwood in [10].

We begin by first giving some fundamental concepts, definitions and propositions that will be used in this paper. The number of maximal chains of the finite abelian group $p^n q^m r$ is given in [7] and briefly discussed here. Using the equivalence relation given by Murali and Makamba in [3] and their criss-cut counting technique therein, we classify the fuzzy subgroups of the abelian group

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$p^n q^m r$ for $m = 1, 2, 3, 4$. This will extend the work of [3] where a classification of the fuzzy subgroups of the finite abelian group $p^n q^m$ was done using the cross-cut counting technique.

2. Preliminaries

Since our counting of fuzzy subgroups is anchored on the maximal chains of subgroups of a group G, we look at some important concepts relating to the maximal chains of subgroups of G.

Definition. A proper subgroup M of a group G is called *maximal* if whenever $M \leq H \leq G$, then either $H = M$ or $H = G$. A chain of subgroups of a group is said to be a *maximal chain* if it cannot be properly contained in another chain.

O. Ndiweni in [7], working on the number of maximal chains of finite abelian groups, gave the following results in Proposition 2.1.

Proposition 2.1. *The group* $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^m} \times \mathbb{Z}_{r^s}$ *has* $\frac{(n+m+s)!}{n!m!s!}$ *maximal chains.*

Remark 2.2. The word chain(s) in this paper is used to mean subgroup maximal chain(s) unless otherwise stated.

Let $I = [0, 1]$ be the unit interval of real numbers with the usual ordering and let X be a non-empty set. A fuzzy subset of X is characterized by a function $\mu : X \to I$. μ is called the membership function and $\mu(x)$ is the degree of membership of the element x to the fuzzy subset of X defined by μ.

Definition. The *support* of μ, denoted by *supp*(μ), is defined as *supp*$(\mu) = \{x \in X : \mu(x) > 0\}$.

Definition ([9])**.** Let G be a group. A fuzzy subset μ of G is said to be a fuzzy subgroup of G, if for all $x, y \in G$

(i) $\mu(xy) \geq \min\{m(x), \mu(y)\}$,
(ii) $\mu(x^{-1}) \geq \mu(x)$.

Definition ([3])**.** Two fuzzy subgroups μ and ν of a group G are said to be *equivalent* denoted $\mu \sim \nu$ if

(i) for all $x, y \in X$, $\mu(x) > \mu(y)$ if and only if $\nu(x) > \nu(y)$,
(ii) $\mu(x) = 0$ if and only if $\nu(x) = 0$.

Clearly this relation is an equivalence relation on I^X and it coincides with equality of sets when restricted to 2^X.

Definition ([9])**.** Let G be a group. A fuzzy subset μ of G is said to be a fuzzy subgroup of G if for all $x, y \in G$

(i) $\mu(xy) \geq \min\{m(x), \mu(y)\}$,
(ii) $\mu(x^{-1}) \geq \mu(x)$.

Remark 2.3. According to Murali and Makamba [4], two fuzzy subgroups are *distinct* if they are non-equivalent.

In our current work, we use the criss-cut method [6, 8], in counting distinct fuzzy subgroups. In this method, we first list all the maximal chains of the group and then use the counting technique (criss-cut) to enumerate the distinct fuzzy subgroups. The technique is explained in detail in [2, 8], but we give its summarized discussion below.

Remark 2.4. The order of listing our maximal subgroup chains does not matter and so does not alter the number of distinct fuzzy subgroups. Thus we can start the counting from any chain in the list and proceed in any order. Therefore we number our maximal chains here according to the order in which we consider the chains in our counting.

Criss-cut counting technique

Let G be a group having the property that all its maximal chains are of the same length. By length, we mean the number of subgroups in the maximal chain. From the list of the maximal subgroup chains, suppose our first chain is

$$0 \subseteq H_1 \subseteq H_2 \subseteq \cdots \subseteq H_n = G. \tag{1}$$

By [3], the chain (1) contributes $2^{n+1} - 1$ distinct fuzzy subgroups of G. Let our next maximal chain be

$$0 \subseteq J_1 \subseteq J_2 \subseteq \cdots \subseteq J_n = G \tag{2}$$

such that for some i, $J_i \neq H_i$ where $i \in \{1, 2, \dots, n-1\}$. This new subgroup J_i is called a *distinguishing factor* of the maximal chain as it distinguishes the chain from the previous one. If this new chain has two or more subgroups that were not in the first one, we simply pick one and call it a distinguishing factor for the chain. The other new subgroups will be used in subsequent maximal chains as distinguishing factors. Thus if a J_k is new in the second chain but was not used to distinguish that chain from the first, it may be used in the next chosen chain that contains it as a distinguishing factor. However, once used, a new subgroup cannot be a distinguishing factor in another maximal chain. The number of distinct fuzzy subgroups of G contributed by the chain (2) is given by Proposition 2.5.

Proposition 2.5 ([8]). *The number of distinct fuzzy subgroups of G contributed by a maximal subgroup chain with a distinguishing factor is equal to* $\frac{2^{n+1}}{2} = 2^n$ *for* $n \geq 2$.

This process of identifying distinguishing factors is continued until there are no distinguishing factors in the chains. Note that none of the subgroups in the first chain may be used as a distinguishing factor.

After exhausting all distinguishing factors, the next step is to use pairs of subgroups to distinguish maximal chains. Suppose in our counting process, we encounter a maximal subgroup chain $0 \subseteq K_1 \subseteq K_2 \subseteq \cdots \subseteq K_n = G$, such that there are no (single) distinguishing factors, but there is a pair $\{K_i, K_j\}$, $i \neq j$, of subgroups in this chain that have not appeared in any previous chain (together) not necessarily consecutively. We call this pair a *distinguishing pair* for the chain. So if subgroups K_i and K_j have not appeared together in the previous maximal chains, then they are indeed a new pair. As in the case of single distinguishing factors, if there is another new pair (H, K) in that chain containing K_i and K_j, then the pair (H, K) may be used in a subsequent chain as a (new) distinguishing pair. The focus is on using a subgroup or a pair of subgroups to tag (or identify) a maximal chain. The number of (new) distinct fuzzy subgroups contributed by this chain (with a distinguishing pair) is given below.

Proposition 2.6 ([8])**.** *In the process of counting distinct fuzzy subgroups, a maximal subgroup chain that has no single distinguishing factor but has a distinguishing pair, contributes* $\frac{2^{n+1}}{2^2} = 2^{n-1}$ *new distinct fuzzy subgroups of* G *for* $n \geq 4$.

After exhausting all distinguishing pairs, we proceed to distinguishing triples. These are treated just like distinguishing pairs. A chain with no distinguishing factor and no distinguishing pair but has three subgroups (H, J, K) that have not appeared together in any previous chain, has a *distinguishing triple*. Another new triple in the same chain may be used in a subsequent chain as a distinguishing triple. Such a chain contributes $\frac{2^{n+1}}{2^3}$ new distinct fuzzy subgroups.

If the maximal chains have not been exhausted, continue to use a *distinguishing quadruple*. Continue until all the maximal chains have been exhausted. This counting argument can be generalised in Proposition 2.7, which gives only the contribution of a single maximal chain to the total number of distinct fuzzy subgroups other than the first two chosen maximal chains.

Proposition 2.7. *In a finite group* G*, if a maximal subgroup chain of length* $n+1$*, other than the first two chosen chains, has no distinguishing* $(m-1)$*-tuple, but has a distinguishing* m*-tuple for* $m \geq 2$*, then the chain contributes* $\frac{2^{n+1}}{2^m} = 2^{n+1-m}$ *new distinct fuzzy subgroups of* G*,* $n+1 > m$.

Note that a distinguishing 1-tuple is distinguishing factor; a distinguishing 2-tuple is distinguishing pair, and so forth.

Remark 2.8. In this paper, a distinguishing factor is indicated by $*$, a distinguishing pair by $\{*, **\}$ and a distinguishing triple by $\{*, **, ***\}$. This indication can be extended similarly to a distinguishing quadruple and beyond.

Murali and Makamba in [5] worked on the number of distinct fuzzy subgroups of the group $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^m}$ using the cross-cut method. They obtained important results which we summarize in Theorem 2.9.

Theorem 2.9 ([5]). *The number of distinct fuzzy subgroups for the group $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^m}$ is*

$$\left[2^{n+m+1}\sum_{r=0}^{m} 2^{-r}\binom{n}{r}\binom{m}{r}\right] - 1,\ n \geq m.$$

3. Distinct fuzzy subgroups of $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^m} \times \mathbb{Z}_r$, $n, m \in \mathbb{Z}^+$, $m = 1, 2, 3$; p, q, r distinct primes

To achieve our objective of counting the distinct fuzzy subgroups in each group, the maximal chains of each group are listed. The number of distinct fuzzy subgroups is then counted using the criss-cut method of [5]. The counting method used here is different from the one used in [7]. Moreover, the formulae presented here are in polynomial form. The polynomial formulae make it easy for one to see from the coefficients of powers of 2, the number of chains contributing a distinguishing factor, a distinguishing pair, and so on.

3.1. Distinct fuzzy subgroups of $\mathbb{Z}_{p^n} \times \mathbb{Z}_q \times \mathbb{Z}_r$

We begin with the case $m = 1$. When $n = 1, 2$, we list the maximal chains and their corresponding distinct fuzzy subgroups which have been computed manually. For $n = 1, 2$, $\mathbb{Z}_p \times \mathbb{Z}_q \times \mathbb{Z}_r$ and $\mathbb{Z}_{p^2} \times \mathbb{Z}_q \times \mathbb{Z}_r$ have $1 \cdot (2^4 - 1) + 4 \cdot 2^3 + 1 \cdot 2^2 = 51$ and $1 \cdot (2^5 - 1) + 7 \cdot 2^4 + 4 \cdot 2^3 = 175$ distinct fuzzy subgroups respectively. This is shown in Figure 1.

	$p^2qr \supseteq qpr \supseteq pq \supseteq p \supseteq 0 : 2^5 - 1$
	$p^2qr \supseteq qpr \supseteq pq \supseteq q \supseteq 0 : 2^4$
	$p^2qr \supseteq qpr \supseteq pr \supseteq p \supseteq 0 : 2^4$
$pqr \supseteq pq \supseteq p \supseteq 0 : 2^4 - 1$	$p^2qr \supseteq qpr \supseteq pr \supseteq r \supseteq 0 : 2^4$
$pqr \supseteq pq \supseteq q \supseteq 0 : 2^3$	$p^2qr \supseteq qpr \supseteq qr \supseteq q \supseteq 0 : 2^4$
$pqr \supseteq pr \supseteq p \supseteq 0 : 2^3$	$p^2qr \supseteq qpr \supseteq qr \supseteq r \supseteq 0 : 2^3$
$pqr \supseteq pr \supseteq r \supseteq 0 : 2^3$	$p^2qr \supseteq p^2q \supseteq pq \supseteq p \supseteq 0 : 2^4$
$pqr \supseteq qr \supseteq q \supseteq 0 : 2^3$	$p^2qr \supseteq p^2q \supseteq pq \supseteq q \supseteq 0 : 2^3$
$pqr \supseteq qr \supseteq r \supseteq 0 : 2^2$	$p^2qr \supseteq p^2q \supseteq p^2 \supseteq p \supseteq 0 : 2^4$
	$p^2qr \supseteq p^2r \supseteq pr \supseteq p \supseteq 0 : 2^4$
	$p^2qr \supseteq p^2r \supseteq pr \supseteq r \supseteq 0 : 2^3$
	$p^2qr \supseteq p^2r \supseteq p^2 \supseteq p \supseteq 0 : 2^3$

FIGURE 1. Fuzzy subgroups of pqr and p^2qr

For $n = 3, 4$, $\mathbb{Z}_{p^3} \times \mathbb{Z}_q \times \mathbb{Z}_r$ and $\mathbb{Z}_{p^4} \times \mathbb{Z}_q \times \mathbb{Z}_r$ have $1 \cdot (2^6 - 1) + 10 \cdot 2^5 + 9 \cdot 2^4 = 527$ and $1 \cdot (2^7 - 1) + 13 \cdot 2^6 + 16 \cdot 2^5 = 1471$ distinct fuzzy subgroups respectively as shown in Figure 2 and Figure 3.

$$\begin{array}{r}
p^3qr \supseteq qpr \supseteq pq \supseteq p \supseteq 0 : 2^6-1 \\
p^3qr \supseteq p^2qr \supseteq qpr \supseteq pq \supseteq q \supseteq 0 : 2^5 \\
p^3qr \supseteq p^2qr \supseteq qpr \supseteq pr \supseteq p \supseteq 0 : 2^5 \\
p^3qr \supseteq p^2qr \supseteq qpr \supseteq pr \supseteq r \supseteq 0 : 2^5 \\
p^3qr \supseteq p^2qr \supseteq qpr \supseteq qr \supseteq q \supseteq 0 : 2^5 \\
p^3qr \supseteq p^2qr \supseteq qpr \supseteq qr \supseteq r \supseteq 0 : 2^4 \\
p^3qr \supseteq p^2qr \supseteq p^2q \supseteq pq \supseteq p \supseteq 0 : 2^5 \\
p^3qr \supseteq p^2qr \supseteq p^2q \supseteq pq \supseteq q \supseteq 0 : 2^4 \\
p^3qr \supseteq p^2qr \supseteq p^2q \supseteq p^2 \supseteq p \supseteq 0 : 2^5 \\
p^3qr \supseteq p^2qr \supseteq p^2r \supseteq pr \supseteq p \supseteq 0 : 2^5
\end{array}
\underset{Ctd\cdots}{\Rightarrow}
\begin{array}{r}
p^3qr \supseteq p^2qr \supseteq p^2r \supseteq pr \supseteq r \supseteq 0 : 2^4 \\
p^3qr \supseteq p^2qr \supseteq p^2r \supseteq p^2 \supseteq p \supseteq 0 : 2^4 \\
p^3qr \supseteq p^3q \supseteq p^2q \supseteq pq \supseteq p \supseteq 0 : 2^5 \\
p^3qr \supseteq p^3q \supseteq p^2q \supseteq pq \supseteq q \supseteq 0 : 2^4 \\
p^3qr \supseteq p^3q \supseteq p^2q \supseteq p^2 \supseteq p \supseteq 0 : 2^4 \\
p^3qr \supseteq p^3q \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0 : 2^5 \\
p^3qr \supseteq p^3r \supseteq p^2r \supseteq pr \supseteq p \supseteq 0 : 2^5 \\
p^3qr \supseteq p^3r \supseteq p^2r \supseteq pr \supseteq r \supseteq 0 : 2^4 \\
p^3qr \supseteq p^3r \supseteq p^2r \supseteq p^2 \supseteq p \supseteq 0 : 2^4 \\
p^3qr \supseteq p^3r \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0 : 2^4
\end{array}$$

FIGURE 2. Fuzzy subgroups of p^3qr

$$\begin{array}{r}
p^4qr \supseteq p^3qr \supseteq qpr \supseteq pq \supseteq p \supseteq 0 : 2^7-1 \\
p^4qr \supseteq p^3qr \supseteq p^2qr \supseteq qpr \supseteq pq \supseteq q \supseteq 0 : 2^6 \\
p^4qr \supseteq p^3qr \supseteq p^2qr \supseteq qpr \supseteq pr \supseteq p \supseteq 0 : 2^6 \\
p^4qr \supseteq p^3qr \supseteq p^2qr \supseteq qpr \supseteq pr \supseteq r \supseteq 0 : 2^6 \\
p^4qr \supseteq p^3qr \supseteq p^2qr \supseteq qpr \supseteq qr \supseteq q \supseteq 0 : 2^6 \\
p^4qr \supseteq p^3qr \supseteq p^2qr \supseteq qpr \supseteq qr \supseteq r \supseteq 0 : 2^5 \\
p^4qr \supseteq p^3qr \supseteq p^2qr \supseteq p^2q \supseteq pq \supseteq p \supseteq 0 : 2^6 \\
p^4qr \supseteq p^3qr \supseteq p^2qr \supseteq p^2q \supseteq pq \supseteq q \supseteq 0 : 2^5 \\
p^4qr \supseteq p^3qr \supseteq p^2qr \supseteq p^2q \supseteq p^2 \supseteq p \supseteq 0 : 2^6 \\
p^4qr \supseteq p^3qr \supseteq p^2qr \supseteq p^2r \supseteq pr \supseteq p \supseteq 0 : 2^6 \\
p^4qr \supseteq p^3qr \supseteq p^2qr \supseteq p^2r \supseteq pr \supseteq r \supseteq 0 : 2^5 \\
p^4qr \supseteq p^3qr \supseteq p^2qr \supseteq p^2r \supseteq p^2 \supseteq p \supseteq 0 : 2^5 \\
p^4qr \supseteq p^3qr \supseteq p^3q \supseteq p^2q \supseteq pq \supseteq p \supseteq 0 : 2^6 \\
p^4qr \supseteq p^3qr \supseteq p^3q \supseteq p^2q \supseteq pq \supseteq q \supseteq 0 : 2^5 \\
p^4qr \supseteq p^3qr \supseteq p^3q \supseteq p^2q \supseteq p^2 \supseteq p \supseteq 0 : 2^5
\end{array}
\underset{Ctd\cdots}{\Rightarrow}
\begin{array}{r}
p^4qr \supseteq p^3qr \supseteq p^3q \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0 : 2^6 \\
p^4qr \supseteq p^3qr \supseteq p^3r \supseteq p^2r \supseteq pr \supseteq p \supseteq 0 : 2^6 \\
p^4qr \supseteq p^3qr \supseteq p^3r \supseteq p^2r \supseteq pr \supseteq r \supseteq 0 : 2^5 \\
p^4qr \supseteq p^3qr \supseteq p^3r \supseteq p^2r \supseteq p^2 \supseteq p \supseteq 0 : 2^5 \\
p^4qr \supseteq p^3qr \supseteq p^3r \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0 : 2^5 \\
p^4qr \supseteq p^4q \supseteq p^3q \supseteq p^2q \supseteq pq \supseteq p \supseteq 0 : 2^6 \\
p^4qr \supseteq p^4q \supseteq p^3q \supseteq p^2q \supseteq pq \supseteq q \supseteq 0 : 2^5 \\
p^4qr \supseteq p^4q \supseteq p^3q \supseteq p^2q \supseteq p^2 \supseteq p \supseteq 0 : 2^5 \\
p^4qr \supseteq p^4q \supseteq p^3q \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0 : 2^5 \\
p^4qr \supseteq p^4q \supseteq p^4 \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0 : 2^6 \\
p^4qr \supseteq p^4r \supseteq p^3r \supseteq p^2r \supseteq pq \supseteq p \supseteq 0 : 2^6 \\
p^4qr \supseteq p^4r \supseteq p^3r \supseteq p^2r \supseteq pq \supseteq r \supseteq 0 : 2^5 \\
p^4qr \supseteq p^4r \supseteq p^3r \supseteq p^2r \supseteq p^2 \supseteq p \supseteq 0 : 2^5 \\
p^4qr \supseteq p^4r \supseteq p^3r \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0 : 2^5 \\
p^4qr \supseteq p^4r \supseteq p^4 \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0 : 2^5
\end{array}$$

FIGURE 3. Fuzzy subgroups of p^4qr

TABLE 1. Fuzzy subgroups of $\mathbb{Z}_{p^n} \times \mathbb{Z}_q \times \mathbb{Z}_r$

n	p^nqr	Number of fuzzy subgroups
1	pqr	$51 = 1 \cdot (2^4 - 1) + 4 \cdot 2^3 + 1 \cdot 2^2$
2	p^2qr	$175 = 1 \cdot (2^5 - 1) + 7 \cdot 2^4 + 4 \cdot 2^3$
3	p^3qr	$527 = 1 \cdot (2^6 - 1) + 10 \cdot 2^5 + 9 \cdot 2^4$
4	p^4qr	$1471 = 1 \cdot (2^7 - 1) + 13 \cdot 2^6 + 16 \cdot 2^5$
5	p^5qr	$3903 = 1 \cdot (2^8 - 1) + 16 \cdot 2^7 + 25 \cdot 2^6$
6	p^6qr	$9983 = 1 \cdot (2^9 - 1) + 19 \cdot 2^8 + 36 \cdot 2^7$
7	p^7qr	$24831 = 1 \cdot (2^{10} - 1) + 22 \cdot 2^9 + 49 \cdot 2^8$
$\vdots$	$\vdots$	$\vdots$
k	p^kqr	$2^{k+3} - 1 + (3k+1) \cdot 2^{k+2} + k^2 \cdot 2^{k+1}$

Table 1, which extends these results to higher values of n, suggests:

Proposition 3.1. *The number of distinct fuzzy subgroups of the group* $\mathbb{Z}_{p^n} \times \mathbb{Z}_q \times \mathbb{Z}_r$ *is*

$$2^{n+3} - 1 + (3n+1) \cdot 2^{n+2} + n^2 \cdot 2^{n+1}.$$

Proof. We proceed by induction on n. For $n = 1$, $\mathbb{Z}_p \times \mathbb{Z}_q \times \mathbb{Z}_r$ has 6 maximal chains as computed in the discussion that lead to Table 1.

The first maximal chain yields $2^4 - 1$ distinct fuzzy subgroups. Each of the next 4 chains has a distinguishing factor, thus contributes 2^3 distinct fuzzy subgroups. The last chain has a new pair and therefore contributes 2^2 distinct fuzzy subgroups. Hence $\mathbb{Z}_p \times \mathbb{Z}_q \times \mathbb{Z}_r$ has $2^4 - 1 + 4 \cdot 2^3 + 2^2$ distinct fuzzy subgroups. Clearly this number is also obtainable by letting $n = 1$ in the formula of Proposition 3.1. Thus the proposition is true for $n = 1$.

Now assume $\mathbb{Z}_{p^k} \times \mathbb{Z}_q \times \mathbb{Z}_r$ has $2^{k+3} - 1 + (3k+1) \cdot 2^{k+2} + k^2 \cdot 2^{k+1}$ distinct fuzzy subgroups. We want to show that $\mathbb{Z}_{p^{k+1}} \times \mathbb{Z}_q \times \mathbb{Z}_r$ has $2^{k+4} - 1 + [3(k+1)+1] \cdot 2^{k+3} + (k+1)^2 \cdot 2^{k+2}$ distinct fuzzy subgroups. Let $G = \mathbb{Z}_{p^{k+1}} \times \mathbb{Z}_q \times \mathbb{Z}_r$. The maximal chains of G have length $k+4$. These maximal chains are shown in Figure 4.

$$p^{k+1}qr \supseteq p^kqr \supseteq p^{k-1}qr \supseteq \begin{cases} p^{k-2}qr \supseteq \begin{cases} \cdots \\ \cdots \\ \cdots \end{cases} \\ p^{k-1}q \supseteq \begin{cases} \cdots \\ \cdots \end{cases} \\ p^{k-1}r \supseteq \begin{cases} \cdots \\ \cdots \end{cases} \end{cases}$$

$$p^{k+1}qr \supseteq p^kqr \supseteq p^kq \supseteq \begin{cases} p^{k-1}q \supseteq \begin{cases} \cdots \\ \cdots \end{cases} \\ p^k \supseteq \cdots \end{cases}$$

$$p^{k+1}qr \supseteq p^kqr \supseteq p^kr \supseteq \begin{cases} p^{k-1}r \supseteq \begin{cases} \cdots \\ \cdots \end{cases} \\ p^k \supseteq \cdots \end{cases}$$

$$\underset{Ctd\cdots}{\Rightarrow}$$

$$p^{k+1}qr \supseteq p^{k+1}q \supseteq p^kq \supseteq \begin{cases} p^{k-1}q \supseteq \begin{cases} \cdots \\ \cdots \end{cases} \\ p^k \supseteq \cdots \end{cases}$$

$$p^{k+1}qr \supseteq p^{k+1}q \supseteq p^{k+1} \supseteq p^k \supseteq p^{k-1} \cdots \supseteq 0$$

$$p^{k+1}qr \supseteq p^{k+1}r \supseteq p^kr \supseteq \begin{cases} p^{k-1}r \supseteq \begin{cases} \cdots \\ \cdots \end{cases} \\ p^k \supseteq \cdots \end{cases}$$

$$p^{k+1}qr \supseteq p^{k+1}r \supseteq p^{k+1} \supseteq p^k \supseteq p^{k-1} \cdots \supseteq 0$$

FIGURE 4. Maximal chains of $p^{k+1}qr$

The group G has 3 maximal subgroups $H_1 = p^kqr$, $H_2 = p^{k+1}q$ and $H_3 = p^{k+1}r$ from which all the maximal chains extend. So we proceed along these three subgroups.

Case (i) : $H_1 = \mathbb{Z}_{p^k} \times \mathbb{Z}_q \times \mathbb{Z}_r$

By Proposition 2.1, this subgroup has $(k+2)(k+1)$ maximal chains. By the inductive hypothesis, H_1 yields $2^{k+4} - 1 + (3k+1) \cdot 2^{k+3} + k^2 \cdot 2^{k+2}$ distinct fuzzy subgroups.

Case (ii) : $H_2 = \mathbb{Z}_{p^{k+1}} \times \mathbb{Z}_q$

The subgroup H_2 has $(k+2)$ maximal chains. It has only 2 chains with a distinguishing factor (in the language of [8]), viz. $p^{k+1}qr \supseteq p^{k+1}q^* \supseteq p^k q \supseteq p^{k-1}q \supseteq \cdots \supseteq pq \supseteq p \supseteq 0$ and $p^{k+1}qr \supseteq p^{k+1}q \supseteq p^{k+1^*} \supseteq p^k \supseteq \cdots \supseteq pq \supseteq p \supseteq 0$, which contribute $2 \cdot 2^{k+3}$ distinct fuzzy subgroups. Each of the remaining k maximal chains along H_2 contributes a distinguishing pair. This accounts for $k \cdot 2^{k+2}$ distinct fuzzy subgroups. Therefore, H_2 yields $2 \cdot 2^{k+3} + k \cdot 2^{k+2}$ distinct fuzzy subgroups G.

<u>*Case* (iii) :</u> $H_3 = \mathbb{Z}_{p^{k+1}} \times \mathbb{Z}_r$

The subgroup H_3 has $(k+2)$ maximal chains and only 1 chain has a distinguishing factor: $p^{k+1}qr \supseteq p^{k+1}r^* \supseteq p^k \supseteq p^{k-1} \supseteq \cdots \supseteq p^2 \supseteq p \supseteq 0$. This contributes 2^{k+3} distinct fuzzy subgroups the group G. Each of the remaining $(k+1)$ maximal chains contribute a distinguishing pair, accounting for $(k+1) \cdot 2^{k+2}$ distinct fuzzy subgroups. Thus, H_3 yields $2^{k+3} + (k+1) \cdot 2^{k+2}$ distinct fuzzy subgroups. Summing up the contributions from case (i)–case (iii), we have

$$\begin{array}{r} 2^{k+4} - 1 + (3k+1) \cdot 2^{k+3} + k^2 \cdot 2^{k+2} \\ +2 \cdot 2^{k+3} + k \cdot 2^{k+2} \\ +1 \cdot 2^{k+3} + (k+1) \cdot 2^{k+2} \\ \hline = 2^{k+4} - 1 + (3k+4) \cdot 2^{k+3} + (k^2 + 2k + 1) \cdot 2^{k+2}. \end{array}$$

Therefore, the number of distinct fuzzy subgroups of $\mathbb{Z}_{p^{k+1}} \times \mathbb{Z}_q \times \mathbb{Z}_r$ is given by $2^{k+4} - 1 + [3(k+1)+1] \cdot 2^{k+3} + (k+1)^2 \cdot 2^{k+2}$. This can also be obtained from the formula $2^{n+3} - 1 + (3n+1) \cdot 2^{n+2} + n^2 \cdot 2^{n+1}$ with $n = k+1$. $\square$

3.2. Distinct fuzzy subgroups of $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^2} \times \mathbb{Z}_r$

For $n = 1$, $\mathbb{Z}_p \times \mathbb{Z}_{q^2} \times \mathbb{Z}_r$ has 12 maximal chains by Proposition 2.1. From Figure 5, it is clear that pq^2r has $1 \cdot (2^5 - 1) + 7 \cdot 2^4 + 4 \cdot 2^3 = 175$ distinct fuzzy subgroups.

$$\begin{array}{rcl} pq^2r \supseteq pqr \supseteq pq \supseteq p \supseteq 0 : 2^5 - 1 & & pq^2r \supseteq pq^2 \supseteq pq \supseteq p \supseteq 0 : 2^4 \\ pq^2r \supseteq pqr \supseteq pq \supseteq q \supseteq 0 : 2^4 & & pq^2r \supseteq pq^2 \supseteq pq \supseteq q \supseteq 0 : 2^3 \\ pq^2r \supseteq pqr \supseteq pr \supseteq p \supseteq 0 : 2^4 & & pq^2r \supseteq pq^2 \supseteq q^2 \supseteq q \supseteq 0 : 2^4 \\ pq^2r \supseteq pqr \supseteq pr \supseteq r \supseteq 0 : 2^4 & \underset{Ctd\cdots}{\Rightarrow} & pq^2r \supseteq q^2r \supseteq qr \supseteq q \supseteq 0 : 2^4 \\ pq^2r \supseteq pqr \supseteq qr \supseteq q \supseteq 0 : 2^4 & & pq^2r \supseteq q^2r \supseteq qr \supseteq r \supseteq 0 : 2^3 \\ pq^2r \supseteq pqr \supseteq qr \supseteq r \supseteq 0 : 2^3 & & pq^2r \supseteq q^2r \supseteq q^2 \supseteq q \supseteq 0 : 2^3 \end{array}$$

FIGURE 5. Fuzzy subgroups of pq^2r

When $n = 2$, $\mathbb{Z}_{p^2} \times \mathbb{Z}_{q^2} \times \mathbb{Z}_r$ has 30 maximal chains and $1 \cdot (2^6 - 1) + 12 \cdot 2^5 + 15 \cdot 2^4 + 2 \cdot 2^3 = 703$ distinct fuzzy subgroups (see Figure 6).

$$\begin{array}{rcl}
p^2q^2r \supseteq p^2qr \supseteq pqr \supseteq pq \supseteq p \supseteq 0 : 2^6-1 & & p^2q^2r \supseteq pq^2r \supseteq pqr \supseteq pr \supseteq r \supseteq 0 : 2^4 \\
p^2q^2r \supseteq p^2qr \supseteq pqr \supseteq pq \supseteq q \supseteq 0 : 2^5 & & p^2q^2r \supseteq pq^2r \supseteq pqr \supseteq qr \supseteq q \supseteq 0 : 2^4 \\
p^2q^2r \supseteq p^2qr \supseteq pqr \supseteq pr \supseteq p \supseteq 0 : 2^5 & & p^2q^2r \supseteq pq^2r \supseteq pqr \supseteq qr \supseteq r \supseteq 0 : 2^3 \\
p^2q^2r \supseteq p^2qr \supseteq pqr \supseteq pr \supseteq r \supseteq 0 : 2^5 & & p^2q^2r \supseteq pq^2r \supseteq q^2r \supseteq qr \supseteq q \supseteq 0 : 2^5 \\
p^2q^2r \supseteq p^2qr \supseteq pqr \supseteq qr \supseteq q \supseteq 0 : 2^5 & & p^2q^2r \supseteq pq^2r \supseteq q^2r \supseteq qr \supseteq r \supseteq 0 : 2^4 \\
p^2q^2r \supseteq p^2qr \supseteq pqr \supseteq qr \supseteq r \supseteq 0 : 2^4 & & p^2q^2r \supseteq pq^2r \supseteq q^2r \supseteq q^2 \supseteq q \supseteq 0 : 2^5 \\
p^2q^2r \supseteq p^2qr \supseteq p^2q \supseteq pq \supseteq p \supseteq 0 : 2^5 & & p^2q^2r \supseteq pq^2r \supseteq pq^2 \supseteq pq \supseteq p \supseteq 0 : 2^5 \\
p^2q^2r \supseteq p^2qr \supseteq p^2q \supseteq pq \supseteq q \supseteq 0 : 2^4 & \underset{Ctd\cdots}{\Rightarrow} & p^2q^2r \supseteq pq^2r \supseteq pq^2 \supseteq pq \supseteq q \supseteq 0 : 2^4 \\
p^2q^2r \supseteq p^2qr \supseteq p^2q \supseteq p^2 \supseteq p \supseteq 0 : 2^5 & & p^2q^2r \supseteq pq^2r \supseteq pq^2 \supseteq p^2 \supseteq p \supseteq 0 : 2^4 \\
p^2q^2r \supseteq p^2qr \supseteq p^2r \supseteq pr \supseteq p \supseteq 0 : 2^5 & & p^2q^2r \supseteq p^2q^2 \supseteq pq^2 \supseteq pq \supseteq p \supseteq 0 : 2^5 \\
p^2q^2r \supseteq p^2qr \supseteq p^2r \supseteq pr \supseteq r \supseteq 0 : 2^4 & & p^2q^2r \supseteq p^2q^2 \supseteq pq^2 \supseteq pq \supseteq q \supseteq 0 : 2^4 \\
p^2q^2r \supseteq p^2qr \supseteq p^2r \supseteq p^2 \supseteq p \supseteq 0 : 2^4 & & p^2q^2r \supseteq p^2q^2 \supseteq pq^2 \supseteq q^2 \supseteq q \supseteq 0 : 2^4 \\
p^2q^2r \supseteq pq^2r \supseteq pqr \supseteq pq \supseteq p \supseteq 0 : 2^5 & & p^2q^2r \supseteq p^2q^2 \supseteq p^2q \supseteq pq \supseteq p \supseteq 0 : 2^4 \\
p^2q^2r \supseteq pq^2r \supseteq pqr \supseteq pq \supseteq q \supseteq 0 : 2^4 & & p^2q^2r \supseteq p^2q^2 \supseteq p^2q \supseteq pq \supseteq q \supseteq 0 : 2^3 \\
p^2q^2r \supseteq pq^2r \supseteq pqr \supseteq pr \supseteq p \supseteq 0 : 2^4 & & p^2q^2r \supseteq p^2q^2 \supseteq p^2q \supseteq p^2 \supseteq p \supseteq 0 : 2^4
\end{array}$$

FIGURE 6. Fuzzy subgroups of p^2q^2r

This process can similarly be extended to $p^3q^2r,\ p^4q^2r, \dots, p^7q^2r$ to get the results of Table 2.

TABLE 2. Fuzzy subgroups of $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^2} \times \mathbb{Z}_r$

n	p^nq^2r	Number of fuzzy subgroups
1	pq^2r	$175 = 1 \cdot (2^5-1) + 7 \cdot 2^4 + 4 \cdot 2^3$
2	p^2q^2r	$703 = 1 \cdot (2^6-1) + 12 \cdot 2^5 + 15 \cdot 2^4 + 2 \cdot 2^3$
3	p^3q^2r	$2415 = 1 \cdot (2^7-1) + 17 \cdot 2^6 + 33 \cdot 2^5 + 9 \cdot 2^4$
4	p^4q^2r	$7551 = 1 \cdot (2^8-1) + 22 \cdot 2^7 + 58 \cdot 2^6 + 24 \cdot 2^5$
5	p^5q^2r	$22143 = 1 \cdot (2^9-1) + 27 \cdot 2^8 + 90 \cdot 2^7 + 50 \cdot 2^6$
6	p^6q^2r	$61951 = 1 \cdot (2^{10}-1) + 32 \cdot 2^9 + 129 \cdot 2^8 + 90 \cdot 2^7$
7	p^7q^2r	$167167 = 1 \cdot (2^{11}-1) + 37 \cdot 2^{10} + 175 \cdot 2^9 + 147 \cdot 2^8$
$\vdots$	$\vdots$	$\vdots$
k	p^kq^2r	$2^{k+4} - 1 + (2+5k) \cdot 2^{k+3} + \left(\frac{7k^2+k}{2}\right) \cdot 2^{k+2} + \left[\frac{k^2(k-1)}{2}\right] \cdot 2^{k+1}$

This discussion suggests that the number of distinct fuzzy subgroups of the finite abelian group p^nq^2r is $2^{n+4} - 1 + (5n+2) \cdot 2^{n+3} + \left(\frac{7n^2+n}{2}\right) \cdot 2^{n+2} + \left[\frac{n^2(n-1)}{2}\right] \cdot 2^{n+1}$, which we state in Proposition 3.2.

Proposition 3.2. *The number of distinct fuzzy subgroups of the group $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^2} \times \mathbb{Z}_r$ is*

$$2^{n+4} - 1 + (5n+2) \cdot 2^{n+3} + \left(\frac{7n^2+n}{2!}\right) \cdot 2^{n+2} + \left[\frac{n^2(n-1)}{2!}\right] \cdot 2^{n+1}.$$

Proof. We proceed inductively on n. When $n = 1$, $\mathbb{Z}_p \times \mathbb{Z}_{q^2} \times \mathbb{Z}_r \cong \mathbb{Z}_{p^2} \times \mathbb{Z}_q \times \mathbb{Z}_r$ has $2^5 - 1 + 7 \cdot 2^4 + 4 \cdot 2^3$ distinct fuzzy subgroups by Proposition 3.1. This number can clearly be obtained by the substitution of $n = 1$ in the formula of Proposition 3.2.

Suppose the result holds for $n = k$, i.e., $\mathbb{Z}_{p^k} \times \mathbb{Z}_{q^2} \times \mathbb{Z}_r$ has $2^{k+4} - 1 + (5k+2) \cdot 2^{k+3} + \frac{7k^2+k}{2!} \cdot 2^{k+2} + \frac{k^2(k-1)}{2!} \cdot 2^{k+1}$ distinct fuzzy subgroups. We need to show that $G = \mathbb{Z}_{p^{k+1}} \times \mathbb{Z}_{q^2} \times \mathbb{Z}_r$ has $2^{k+5} - 1 + [(5(k+1)+2] \cdot 2^{k+4} + \frac{7(k+1)^2+(k+1)}{2!} \cdot 2^{k+3} + \frac{(k+1)^2 k}{2!} \cdot 2^{k+2}$ distinct fuzzy subgroups. The group G has 3 maximal subgroups $H_1 = p^k q^2 r$, $H_2 = p^{k+1} qr$ and $H_3 = p^{k+1} q^2$ through which all the maximal chains of G pass. These $\frac{(k+r+3)!}{(k+1)!r!2!}$ chains are sketched below:

$$p^{k+1}q^2 r \supseteq p^k q^2 r \supseteq \begin{cases} \cdots \\ \cdots \\ \cdots \end{cases},$$

$$p^{k+1}q^2 r \supseteq p^{k+1} qr \supseteq \begin{cases} \cdots \\ \cdots \\ \cdots \end{cases} \quad \text{and}$$

$$p^{k+1}q^2 r \supseteq p^{k+1} q^2 \supseteq \begin{cases} \cdots \\ \cdots \end{cases}$$

Case (i) : $H_1 = \mathbb{Z}_{p^k} \times \mathbb{Z}_{q^2} \times \mathbb{Z}_r$

By the inductive hypothesis, H_1 has $2^{k+5} - 1 + (5k+2) \cdot 2^{k+4} + \frac{7k^2+k}{2!} \cdot 2^{k+3} + \frac{k^2(k-1)}{2!} \cdot 2^{k+2}$ distinct fuzzy subgroups

Case (ii) : $H_2 = \mathbb{Z}_{p^{k+1}} \times \mathbb{Z}_q \times \mathbb{Z}_r$

There are 4 maximal chains with a distinguishing factor along this subgroup illustrated in Figure 7.

$$\begin{aligned}
p^{k+1}q^2 r \supseteq p^{k+1}qr^* \supseteq p^k qr \supseteq p^{k-1}qr \supseteq p^{k-2}qr \supseteq \cdots \supseteq pqr \supseteq pq \supseteq p \supseteq 0 \\
p^{k+1}q^2 r \supseteq p^{k+1}qr \supseteq p^{k+1}q^* \supseteq p^k q \supseteq p^{k-1}q \supseteq \cdots \supseteq p^2 q \supseteq pq \supseteq p \supseteq 0 \\
p^{k+1}q^2 r \supseteq p^{k+1}qr \supseteq p^{k+1}q \supseteq p^{k+1^*} \supseteq p^{k-1} \supseteq \cdots \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0 \\
p^{k+1}q^2 r \supseteq p^{k+1}qr \supseteq p^{k+1}r^* \supseteq p^k r \supseteq p^{k-1}r \supseteq \cdots \supseteq p^2 r \supseteq pr \supseteq p \supseteq 0
\end{aligned}$$

FIGURE 7. Maximal chains of $H_2 = p^{k+1}qr$ with a distinguishing factor

The subgroup H_2 has 5 clusters shown in Figures 8–12 which have respectively $k,\ k,\ (k+1),\ k,\ (k+1)$ maximal chains with a distinguishing pair.

$$
\begin{array}{c}
p^{k+1}q^2r \supseteq p^{k+1}qr^* \supseteq p^kqr \supseteq p^{k-1}qr \supseteq \cdots \supseteq p^3qr \supseteq p^2qr \supseteq pqr \supseteq pq^{**} \supseteq q \supseteq 0\\
p^{k+1}q^2r \supseteq p^{k+1}qr^* \supseteq p^kqr \supseteq p^{k-1}qr \supseteq \cdots \supseteq p^3qr \supseteq p^2qr \supseteq p^2q^{**} \supseteq pq \supseteq p \supseteq 0\\
p^{k+1}q^2r \supseteq p^{k+1}qr^* \supseteq p^kqr \supseteq p^{k-1}qr \supseteq \cdots \supseteq p^3qr \supseteq p^3q^{**} \supseteq p^2q \supseteq pq \supseteq p \supseteq 0\\
\vdots \qquad\qquad \vdots \qquad\qquad \vdots\\
p^{k+1}q^2r \supseteq p^{k+1}qr^* \supseteq p^kqr \supseteq p^kq^{**} \supseteq p^{k-1}q \supseteq \cdots \supseteq p^3q \supseteq p^2q \supseteq pq \supseteq p \supseteq 0
\end{array}
$$

FIGURE 8. First cluster of k chains of $H_2 = p^{k+1}qr$ with a distinguishing pair

$$
\begin{array}{c}
p^{k+1}q^2r \supseteq p^{k+1}qr^* \supseteq p^kqr \supseteq p^{k-1}qr \supseteq \cdots \supseteq p^3qr \supseteq p^2qr \supseteq pqr \supseteq pr^{**} \supseteq q \supseteq 0\\
p^{k+1}q^2r \supseteq p^{k+1}qr^* \supseteq p^kqr \supseteq p^{k-1}qr \supseteq \cdots \supseteq p^3qr \supseteq p^2qr \supseteq p^2r^{**} \supseteq pr \supseteq p \supseteq 0\\
p^{k+1}q^2r \supseteq p^{k+1}qr^* \supseteq p^kqr \supseteq p^{k-1}qr \supseteq \cdots \supseteq p^3qr \supseteq p^3r^{**} \supseteq p^2r \supseteq pr \supseteq p \supseteq 0\\
\vdots \qquad\qquad \vdots \qquad\qquad \vdots\\
p^{k+1}q^2r \supseteq p^{k+1}qr^* \supseteq p^kqr \supseteq p^kr^{**} \supseteq p^{k-1}r \supseteq \cdots \supseteq p^3r \supseteq p^2r \supseteq pr \supseteq p \supseteq 0
\end{array}
$$

FIGURE 9. Second cluster of k chains of $H_2 = p^{k+1}qr$ with a distinguishing pair

$$
\begin{array}{c}
p^{k+1}q^2r \supseteq p^{k+1}qr^* \supseteq p^kqr \supseteq p^{k-1}qr \supseteq \cdots \supseteq p^3qr \supseteq p^2qr \supseteq pqr \supseteq qr^{**} \supseteq q \supseteq 0\\
p^{k+1}q^2r \supseteq p^{k+1}qr^* \supseteq p^kqr \supseteq p^{k-1}qr \supseteq \cdots \supseteq p^3qr \supseteq p^2qr \supseteq pqr \supseteq qr \supseteq r^{**} \supseteq 0\\
p^{k+1}q^2r \supseteq p^{k+1}qr^* \supseteq p^kqr \supseteq p^{k-1}qr \supseteq \cdots \supseteq p^3qr \supseteq p^2qr \supseteq p^2q \supseteq p^{2^{**}} \supseteq p \supseteq 0\\
p^{k+1}q^2r \supseteq p^{k+1}qr^* \supseteq p^kqr \supseteq p^{k-1}qr \supseteq \cdots \supseteq p^3qr \supseteq p^3q \supseteq p^{3^{**}} \supseteq p^2 \supseteq p \supseteq 0\\
\vdots \qquad\qquad \vdots \qquad\qquad \vdots\\
p^{k+1}q^2r \supseteq p^{k+1}qr^* \supseteq p^kqr \supseteq p^kq \supseteq p^{k^{**}} \supseteq p^{k-1} \cdots \supseteq p^4 \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0
\end{array}
$$

FIGURE 10. Third cluster of $k+1$ chains of $H_2 = p^{k+1}qr$ with a distinguishing pair

$$
\begin{array}{c}
p^{k+1}q^2r \supseteq p^{k+1}qr \supseteq p^{k+1}q^* \supseteq p^kq \supseteq p^{k-1}q \supseteq \cdots \supseteq p^3q \supseteq p^2q \supseteq pq \supseteq q^{**} \supseteq 0\\
p^{k+1}q^2r \supseteq p^{k+1}qr \supseteq p^{k+1}q^* \supseteq p^kq \supseteq p^{k-1}q \supseteq \cdots \supseteq p^3q \supseteq p^2q \supseteq p^{2^{**}} \supseteq p \supseteq 0\\
p^{k+1}q^2r \supseteq p^{k+1}qr \supseteq p^{k+1}q^* \supseteq p^kq \supseteq p^{k-1}q \supseteq \cdots \supseteq p^3q \supseteq p^{3^{**}} \supseteq p^2 \supseteq p \supseteq 0\\
\vdots \qquad\qquad \vdots \qquad\qquad \vdots\\
p^{k+1}q^2r \supseteq p^{k+1}qr \supseteq p^{k+1}q^* \supseteq p^kq \supseteq p^{k^{**}} \supseteq p^{k-1} \supseteq \cdots \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0
\end{array}
$$

FIGURE 11. Fourth cluster of k chains of $H_2 = p^{k+1}qr$ with a distinguishing pair

$$p^{k+1}q^2r \supseteq p^{k+1}qr \supseteq p^{k+1}r^* \supseteq p^kr \supseteq p^{k-1}r \supseteq \cdots \supseteq p^3r \supseteq p^2r \supseteq pr \supseteq r^{**} \supseteq 0$$
$$p^{k+1}q^2r \supseteq p^{k+1}qr \supseteq p^{k+1}r^* \supseteq p^kr \supseteq p^{k-1}r \supseteq \cdots \supseteq p^3r \supseteq p^2r \supseteq p^{2^{**}} \supseteq p \supseteq 0$$
$$p^{k+1}q^2r \supseteq p^{k+1}qr \supseteq p^{k+1}r^* \supseteq p^kr \supseteq p^{k-1}r \supseteq \cdots \supseteq p^3r \supseteq p^{3^{**}} \supseteq p^2 \supseteq p \supseteq 0$$
$$\vdots \qquad \vdots \qquad \vdots$$
$$p^{k+1}q^2r \supseteq p^{k+1}qr \supseteq p^{k+1}r^* \supseteq p^kr \supseteq p^{k^{**}} \supseteq p^{k-1} \supseteq \cdots \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0$$
$$p^{k+1}q^2r \supseteq p^{k+1}qr \supseteq p^{k+1}r^* \supseteq p^{k+1^{**}} \supseteq p^k \supseteq p^{k-1} \supseteq \cdots \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0$$

FIGURE 12. Fifth cluster of $k+1$ chains of $H_2 = p^{k+1}qr$ with a distinguishing pair

From these 5 clusters, we have that H_2 has a total of $(5k+2)$ maximal chains with a distinguishing pair.

Similarly, H_2 has clusters of chains with a distinguishing triple. Each of these clusters has k chains which we enumerate as follows: The first cluster has chains ending with $pr \supseteq r \supseteq 0$, the second ends with $pq \supseteq q \supseteq 0$, the third, $p^2 \supseteq p \supseteq 0$, the fourth, $p^3 \supseteq p^2 \supseteq p \supseteq p \supseteq 0$ and so on, with the last cluster comprising of chains ending with $p^{k-1} \supseteq p^{k-2} \supseteq \cdots \supseteq p^2 \supseteq p \supseteq p \supseteq 0$. This gives a total of k clusters. Hence H_2 has k^2 maximal chains with a distinguishing triple.

Case (iii) : $H_3 = \mathbb{Z}_{p^{k+1}} \times \mathbb{Z}_{q^2}$

There is only 1 maximal chain with a distinguishing factor through this subgroup. This is the chain $p^{k+1}q^2r \supseteq p^{k+1}q^{2^*} \supseteq p^kq^2 \supseteq p^{k-1}q^2 \supseteq p^{k-2}q^2 \supseteq \cdots \supseteq pq^2 \supseteq pq \supseteq p \supseteq 0$. Extending through H_3, we have 2 clusters of maximal chains with a distinguishing pair. The first cluster (in Figure 13) has $(k+2)$ maximal chains. The second cluster has k maximal chains from Figure 14. Therefore, the subgroup H_3 yields $2k+2$ maximal chains with a distinguishing pair.

$$p^{k+1}q^2r \supseteq p^{k+1}q^{2^*} \supseteq p^kq^2 \supseteq p^{k-1}q^2 \supseteq p^{k-2}q^2 \supseteq \cdots \supseteq p^2q^2 \supseteq pq^2 \supseteq pq \supseteq q^{**} \supseteq 0$$
$$p^{k+1}q^2r \supseteq p^{k+1}q^{2^*} \supseteq p^kq^2 \supseteq p^{k-1}q^2 \supseteq p^{k-2}q^2 \supseteq \cdots \supseteq p^2q^2 \supseteq pq^2 \supseteq q^{2^{**}} \supseteq q \supseteq 0$$
$$p^{k+1}q^2r \supseteq p^{k+1}q^{2^*} \supseteq p^kq^2 \supseteq p^{k-1}q^2 \supseteq p^{k-2}q^2 \supseteq \cdots \supseteq p^2q^2 \supseteq p^2q^{**} \supseteq pq \supseteq p \supseteq 0$$
$$p^{k+1}q^2r \supseteq p^{k+1}q^{2^*} \supseteq p^kq^2 \supseteq p^{k-1}q^2 \supseteq \cdots \supseteq p^3q^2 \supseteq p^3q^{**} \supseteq p^2q \supseteq pq \supseteq p \supseteq 0$$
$$\vdots \qquad \vdots \qquad \vdots$$
$$p^{k+1}q^2r \supseteq p^{k+1}q^{2^*} \supseteq p^kq^2 \supseteq p^kq^{**} \supseteq p^{k-1}q \supseteq \cdots \supseteq p^3q \supseteq p^2q \supseteq pq \supseteq p \supseteq 0$$
$$p^{k+1}q^2r \supseteq p^{k+1}q^{2^*} \supseteq p^{k+1}q^{**} \supseteq p^kq \supseteq p^{k-1}q \supseteq \cdots \supseteq p^3q \supseteq p^2q \supseteq pq \supseteq p \supseteq 0$$

FIGURE 13. First cluster of $k+2$ chains of $H_3 = p^{k+1}q^2$ with a distinguishing pair

$$
\begin{gathered}
p^{k+1}q^2r \supseteq p^{k+1}q^{2^*} \supseteq p^kq^2 \supseteq p^{k-1}q^2 \supseteq \cdots \supseteq p^4q^2 \supseteq p^3q^2 \supseteq p^2q^2 \supseteq p^2q^{**} \supseteq p^2 \supseteq p \supseteq 0 \\
p^{k+1}q^2r \supseteq p^{k+1}q^{2^*} \supseteq p^kq^2 \supseteq p^{k-1}q^2 \supseteq \cdots \supseteq p^4q^2 \supseteq p^3q^2 \supseteq p^3q^{**} \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0 \\
p^{k+1}q^2r \supseteq p^{k+1}q^{2^*} \supseteq p^kq^2 \supseteq p^{k-1}q^2 \supseteq \cdots \supseteq p^4q^2 \supseteq p^4q^{**} \supseteq p^4 \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0 \\
\vdots \qquad\qquad \vdots \qquad\qquad \vdots \\
p^{k+1}q^2r \supseteq p^{k+1}q^{2^*} \supseteq p^kq^2 \supseteq p^kq^{**} \supseteq p^k \supseteq p^{k-1} \supseteq p^{k-2} \supseteq \cdots \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0 \\
p^{k+1}q^2r \supseteq p^{k+1}q^{2^*} \supseteq p^{k+1}q^{**} \supseteq p^{k+1} \supseteq p^k \supseteq p^{k-1} \supseteq p^{k-2} \supseteq \cdots \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0
\end{gathered}
$$

FIGURE 14. Second cluster of k chains of $H_3 = p^{k+1}q^2$ with a distinguishing pair

Next we look at the maximal chains through H_3 contributing a distinguishing triple. Following similar criteria, we have two major clusters of maximal chains with specific patterns. The first cluster has k (Figure 15) maximal chains with a distinguishing triple. All of these first cluster maximal chains end with $pq \supseteq q \supseteq 0$. The second cluster consists of maximal chains ending with $p^2 \supseteq p \supseteq 0$ and is broken down into subclusters in Figures 16–19. So far, the first, second, third and fourth subclusters have respectively 3, 3, 4 and 5 maximal chains. Similarly the fifth, sixth and seventh (which we have not included here) subclusters have respectively 6, 7, and 8 maximal chains. In total there are $(k-2)$ subclusters and the last subcluster, shown in Figure 20 has $(k-1)$ maximal chains.

$$
\begin{gathered}
p^{k+1}q^2r \supseteq p^{k+1}q^{2^*} \supseteq p^kq^{2^{**}} \supseteq p^{k-1}q^2 \supseteq \cdots \supseteq p^4q^2 \supseteq p^3q^2 \supseteq p^2q^2 \supseteq p^2q^{***} \supseteq pq \supseteq q \supseteq 0 \\
p^{k+1}q^2r \supseteq p^{k+1}q^{2^*} \supseteq p^kq^{2^{**}} \supseteq p^{k-1}q^2 \supseteq \cdots \supseteq p^4q^2 \supseteq p^3q^2 \supseteq p^3q^{***} \supseteq p^2q \supseteq pq \supseteq q \supseteq 0 \\
p^{k+1}q^2r \supseteq p^{k+1}q^{2^*} \supseteq p^kq^{2^{**}} \supseteq p^{k-1}q^2 \supseteq \cdots \supseteq p^4q^2 \supseteq p^4q^{***} \supseteq p^3q \supseteq p^2q \supseteq pq \supseteq q \supseteq 0 \\
p^{k+1}q^2r \supseteq p^{k+1}q^{2^*} \supseteq p^kq^{2^{**}} \supseteq p^{k-1}q^2 \supseteq \cdots \supseteq p^5q^2 \supseteq p^5q^{***} \supseteq p^4q \supseteq p^3q \supseteq p^2q \supseteq pq \supseteq q \supseteq 0 \\
\vdots \qquad\qquad \vdots \qquad\qquad \vdots \\
p^{k+1}q^2r \supseteq p^{k+1}q^{2^*} \supseteq p^kq^{2^{**}} \supseteq p^{k-1}q^2 \supseteq p^{k-1}q^{***} \supseteq \cdots \supseteq p^5q \supseteq p^4q \supseteq p^3q \supseteq p^2q \supseteq pq \supseteq q \supseteq 0 \\
p^{k+1}q^2r \supseteq p^{k+1}q^{2^*} \supseteq p^kq^{2^{**}} \supseteq p^kq^{***} \supseteq p^{k-1}q \supseteq \cdots \supseteq p^5q \supseteq p^4q \supseteq p^3q \supseteq p^2q \supseteq pq \supseteq q \supseteq 0 \\
p^{k+1}q^2r \supseteq p^{k+1}q^{2^*} \supseteq p^{k+1}q^{**} \supseteq p^kq^{***} \supseteq p^{k-1}q \supseteq \cdots \supseteq p^5q \supseteq p^4q \supseteq p^3q \supseteq p^2q \supseteq pq \supseteq q \supseteq 0
\end{gathered}
$$

FIGURE 15. First cluster of k chains of $H_3 = p^{k+1}q^2$ with a distinguishing triple

$$
\begin{gathered}
p^{k+1}q^2r \supseteq p^{k+1}q^{2^*} \supseteq p^kq^2 \supseteq p^{k-1}q^2 \supseteq \cdots \supseteq p^4q^2 \supseteq p^3q^2 \supseteq p^3q^{**} \supseteq p^2q \supseteq p^2 \supseteq p^{***} \supseteq 0 \\
p^{k+1}q^2r \supseteq p^{k+1}q^{2^*} \supseteq p^kq^2 \supseteq p^{k-1}q^2 \supseteq \cdots \supseteq p^4q^2 \supseteq p^4q^{**} \supseteq p^3q \supseteq p^2q \supseteq p^{2^{***}} \supseteq p \supseteq 0 \\
p^{k+1}q^2r \supseteq p^{k+1}q^{2^*} \supseteq p^kq^2 \supseteq p^{k-1}q^2 \supseteq \cdots \supseteq p^4q^2 \supseteq p^4q^{**} \supseteq p^3q \supseteq p^{3^{***}} \supseteq p^2 \supseteq p \supseteq 0
\end{gathered}
$$

FIGURE 16. First subcluster in second cluster of 3 chains of $H_3 = p^{k+1}q^2$ with a distinguishing triple

$$
\begin{array}{r}
p^{k+1}q^2r \supseteq p^{k+1}q^{2^*} \supseteq p^kq^2 \supseteq \cdots \supseteq p^5q^2 \supseteq p^5q^{**} \supseteq p^4q \supseteq p^3q \supseteq p^2q \supseteq p^{2^{***}} \supseteq p \supseteq 0 \\
p^{k+1}q^2r \supseteq p^{k+1}q^{2^*} \supseteq p^kq^2 \supseteq \cdots \supseteq p^5q^2 \supseteq p^5q^{**} \supseteq p^4q \supseteq p^3q \supseteq p^{3^{***}} \supseteq p^2 \supseteq p \supseteq 0 \\
p^{k+1}q^2r \supseteq p^{k+1}q^{2^*} \supseteq p^kq^2 \supseteq \cdots \supseteq p^5q^2 \supseteq p^5q^{**} \supseteq p^4q \supseteq p^{4^{***}} \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0
\end{array}
$$

FIGURE 17. Second subcluster in second cluster of 3 chains of $H_3 = p^{k+1}q^2$ with a distinguishing triple

$$
\begin{array}{r}
p^{k+1}q^2r \supseteq p^{k+1}q^{2^*} \supseteq p^kq^2 \supseteq \cdots \supseteq p^6q^2 \supseteq p^6q^{**} \supseteq p^5q \supseteq p^4q \supseteq p^3q \supseteq p^2q \supseteq p^{2^{***}} \supseteq p \supseteq 0 \\
p^{k+1}q^2r \supseteq p^{k+1}q^{2^*} \supseteq p^kq^2 \supseteq \cdots \supseteq p^6q^2 \supseteq p^6q^{**} \supseteq p^5q \supseteq p^4q \supseteq p^3q \supseteq p^{3^{***}} \supseteq p^2 \supseteq p \supseteq 0 \\
p^{k+1}q^2r \supseteq p^{k+1}q^{2^*} \supseteq p^kq^2 \supseteq \cdots \supseteq p^6q^2 \supseteq p^6q^{**} \supseteq p^5q \supseteq p^4q \supseteq p^{4^{***}} \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0 \\
p^{k+1}q^2r \supseteq p^{k+1}q^{2^*} \supseteq p^kq^2 \supseteq \cdots \supseteq p^6q^2 \supseteq p^6q^{**} \supseteq p^5q \supseteq p^{5^{***}} \supseteq p^4 \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0
\end{array}
$$

FIGURE 18. Third subcluster in second cluster of 4 chains of $H_3 = p^{k+1}q^2$ with a distinguishing triple

$$
\begin{array}{r}
p^{k+1}q^2r \supseteq p^{k+1}q^{2^*} \supseteq \cdots \supseteq p^7q^2 \supseteq p^7q^{**} \supseteq p^6q \supseteq p^5q \supseteq p^4q \supseteq p^3q \supseteq p^2q \supseteq p^{2^{***}} \supseteq p \supseteq 0 \\
p^{k+1}q^2r \supseteq p^{k+1}q^{2^*} \supseteq \cdots \supseteq p^7q^2 \supseteq p^7q^{**} \supseteq p^6q \supseteq p^5q \supseteq p^4q \supseteq p^3q \supseteq p^{3^{***}} \supseteq p^2 \supseteq p \supseteq 0 \\
p^{k+1}q^2r \supseteq p^{k+1}q^{2^*} \supseteq \cdots \supseteq p^7q^2 \supseteq p^7q^{**} \supseteq p^6q \supseteq p^5q \supseteq p^4q \supseteq p^{4^{***}} \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0 \\
p^{k+1}q^2r \supseteq p^{k+1}q^{2^*} \supseteq \cdots \supseteq p^7q^2 \supseteq p^7q^{**} \supseteq p^6q \supseteq p^5q \supseteq p^{5^{***}} \supseteq p^4 \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0 \\
p^{k+1}q^2r \supseteq p^{k+1}q^{2^*} \supseteq \cdots \supseteq p^7q^2 \supseteq p^7q^{**} \supseteq p^6q \supseteq p^{6^{***}} \supseteq p^5 \supseteq p^4 \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0
\end{array}
$$

FIGURE 19. Fourth subcluster in second cluster of 5 chains of $H_3 = p^{k+1}q^2$ with a distinguishing triple

$$
\begin{array}{r}
p^{k+1}q^2r \supseteq p^{k+1}q^{2^*} \supseteq p^{k+1}q^{**} \supseteq p^kq \supseteq p^{k-1}q \supseteq \cdots \supseteq p^5q \supseteq p^4q \supset p^3q \supseteq p^2q \supseteq p^{2^{***}} \supseteq p \supseteq 0 \\
p^{k+1}q^2r \supseteq p^{k+1}q^{2^*} \supseteq p^{k+1}q^{**} \supseteq p^kq \supseteq p^{k-1}q \supseteq \cdots \supseteq p^5q \supseteq p^4q \supset p^3q \supseteq p^{3^{***}} \supseteq p^2 \supseteq p \supseteq 0 \\
p^{k+1}q^2r \supseteq p^{k+1}q^{2^*} \supseteq p^{k+1}q^{**} \supseteq p^kq \supseteq p^{k-1}q \supseteq \cdots \supseteq p^5q \supseteq p^4q \supset p^{4^{***}} \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0 \\
p^{k+1}q^2r \supseteq p^{k+1}q^{2^*} \supseteq p^{k+1}q^{**} \supseteq p^kq \supseteq p^{k-1}q \supseteq \cdots \supseteq p^5q \supseteq p^{5^{***}} \supseteq p^4 \supset p^3 \supseteq p^2 \supseteq p \supseteq 0 \\
\vdots \qquad\qquad\qquad \vdots \qquad\qquad\qquad \vdots \\
p^{k+1}q^2r \supseteq p^{k+1}q^{2^*} \supseteq p^{k+1}q^{**} \supseteq p^kq \supseteq p^{k-1}q \supseteq p^{k-1^{***}} \supseteq \cdots \supseteq p^5 \supseteq p^4 \supset p^3 \supseteq p^2 \supseteq p \supseteq 0 \\
p^{k+1}q^2r \supseteq p^{k+1}q^{2^*} \supseteq p^{k+1}q^{**} \supseteq p^kq \supseteq p^{k^{***}} \supseteq p^{k-1} \supseteq \cdots \supseteq p^5 \supseteq p^4 \supset p^3 \supseteq p^2 \supseteq p \supseteq 0
\end{array}
$$

FIGURE 20. The $k-2^{nd}$ subcluster in second cluster of $k-1$ chains of $H_3 = p^{k+1}q^2$ with a distinguishing triple

The number of maximal chains in the subclusters of the second cluster is summarized in Table 3.

TABLE 3. Maximal chains of $H_3 = p^{m+1}q^2$ with a distinguishing triple in each subcluster of the second cluster

S-cl \ m	3	4	5	6	7	8	$\cdots$	k
1	3	3	3	3	3	3	$\cdots$	3
2		3	3	3	3	3	$\cdots$	3
3			4	4	4	4	$\cdots$	4
4				5	5	5	$\cdots$	5
5					6	6	$\cdots$	6
6						7	$\cdots$	7
$\vdots$							$\ddots$	$\vdots$
$k-2$								$k-1$

In Table 3, S-cl means subcluster. When $m = k$, each of the first two subclusters has 3 maximal chains. From the third subcluster onwards, the number of chains form an arithmetic sequence $4, 5, 6, 7, 8, \ldots, (k-1)$. This sequence has $k-4$ terms with the first term $a = 4$, common difference $d = 1$ and last term $l = k-1$. Thus, the number of maximal chains from the third subcluster is $S_n = \frac{n}{2}(a+l) = \frac{(k-4)(k+3)}{2}$. Therefore the total number of maximal chains with a distinguishing triple contributed by H_3 is $\frac{(k-4)(k+3)}{2!} + (6+k) = \frac{k(k+1)}{2!}$.

Summing up the contributions from case (i)-case (iii), we get $2^{k+5} - 1 + (5k+7) \cdot 2^{k+4} + \frac{7k^2+15k+8}{2!} \cdot 2^{k+3} + \frac{k(k^2+2k+1)}{2!} \cdot 2^{k+2}$. Hence the group $\mathbb{Z}_{p^{k+1}} \times \mathbb{Z}_{q^2} \times \mathbb{Z}_r$ has $2^{k+5} + [(5(k+1)+2)+2] \cdot 2^{k+4} + \frac{7(k+1)^2+(k+1)}{2!} \cdot 2^{k+3} + \frac{(k+1)^2 k}{2!} \cdot 2^{k+2}$ distinct fuzzy subgroups. This result can also be obtained by substituting $n = k+1$ in the formula $2^{n+4} - 1 + (5n+2) \cdot 2^{n+3} + \left(\frac{7n^2+n}{2!}\right) \cdot 2^{n+2} + \left[\frac{n^2(n-1)}{2!}\right] \cdot 2^{n+1}$. $\square$

3.3. Distinct fuzzy subgroups of $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^3} \times \mathbb{Z}_r$

As in Subsections 3.1 and 3.2, distinct fuzzy subgroups for various values of n and $q = 3$ fixed were computed. The results are reflected in Table 4.

TABLE 4. Fuzzy subgroups of $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^3} \times \mathbb{Z}_r$

n	$p^n q^3 r$	Number of fuzzy subgroups
1	pq^3r	$527 = 1 \cdot (2^6 - 1) + 10 \cdot 2^5 + 9 \cdot 2^4$
2	p^2q^3r	$2415 = 1 \cdot (2^7 - 1) + 17 \cdot 2^6 + 33 \cdot 2^5 + 9 \cdot 2^4$
3	p^3q^3r	$9263 = 1 \cdot (2^8 - 1) + 24 \cdot 2^7 + 72 \cdot 2^6 + 40 \cdot 2^5 + 3 \cdot 2^4$
4	p^4q^3r	$31871 = 1 \cdot (2^9 - 1) + 31 \cdot 2^8 + 126 \cdot 2^7 + 106 \cdot 2^6 + 16 \cdot 2^5$
5	p^5q^3r	$101759 = 1 \cdot (2^{10} - 1) + 38 \cdot 2^9 + 195 \cdot 2^8 + 220 \cdot 2^7 + 50 \cdot 2^6$
6	p^6q^3r	$307445 = 1 \cdot (2^{11} - 1) + 45 \cdot 2^{10} + 279 \cdot 2^9 + 395 \cdot 2^8 + 120 \cdot 2^7$
7	p^7q^3r	$890111 = 1 \cdot (2^{12} - 1) + 52 \cdot 2^{11} + 378 \cdot 2^{10} + 644 \cdot 2^9 + 245 \cdot 2^8$
$\vdots$	$\vdots$	$\vdots$
k	p^kq^3r	$2^{k+5} - 1 + (7k+3) \cdot 2^{k+4} + \frac{3(5k^2+k)}{2} \cdot 2^{k+3} + (2k^3 - k^2 - 2k + 2) \cdot 2^{k+2} + \frac{k^4 - 2k^3 - 4k^2 + 11k - 6}{3!} \cdot 2^{k+1}$

The intricacies involved in the construction of the above table are unpacked in the following proposition.

Proposition 3.3. *The number of distinct fuzzy subgroups of the group* $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^3} \times \mathbb{Z}_r$ *is* $2^{n+5} - 1 + (7n+3) \cdot 2^{n+4} + \frac{(15n^2+3n)}{2!} \cdot 2^{n+3} + \frac{(13n^2+n)(n-1)}{3!} \cdot 2^{n+2} + \frac{n^2(n-1)(n-2)}{3!} \cdot 2^{n+1}$.

Proof. The proof is by induction on n. When $n = 1$, $\mathbb{Z}_p \times \mathbb{Z}_{q^3} \times \mathbb{Z}_r \cong \mathbb{Z}_{p^3} \times \mathbb{Z}_q \times \mathbb{Z}_r$ has $2^6 - 1 + 10 \cdot 2^5 + 9 \cdot 2^4$ distinct fuzzy subgroups by Proposition 3.1 and 1 of Table 4 . This can also be obtained by substituting $n = 1$ in Proposition 3.3. Similarly, when $n = 2$, $\mathbb{Z}_{p^2} \times \mathbb{Z}_{q^3} \times \mathbb{Z}_r \cong \mathbb{Z}_{p^3} \times \mathbb{Z}_{q^2} \times \mathbb{Z}_r$ has $2^7 - 1 + 17 \cdot 2^6 + 33 \cdot 2^5 + 9 \cdot 2^4$ distinct fuzzy subgroups by Proposition 3.2 and 2 of Table 4.

Suppose the group $\mathbb{Z}_{p^k} \times \mathbb{Z}_{q^3} \times \mathbb{Z}_r$ has $2^{k+5} - 1 + (7k+3) \cdot 2^{k+4} + \frac{(15k^2+3k)}{2!} \cdot 2^{k+3} + \frac{(13k^2+k)(k-1)}{3!} \cdot 2^{k+2} + \frac{k^2(k-1)(k-2)}{3!} \cdot 2^{k+1}$ distinct fuzzy subgroups. We need to show that the group $G = \mathbb{Z}_{p^{k+1}} \times \mathbb{Z}_{q^3} \times \mathbb{Z}_r$ has $2^{k+6} - 1 + [7(k+1) + 3] \cdot 2^{k+5} + \frac{15(k+1)^2+3(k+1)}{2!} \cdot 2^{k+4} + \frac{[13(k+1)^2+(k+1)]k}{3!} \cdot 2^{k+3} + \frac{(k+1)^2k(k-1)}{3!} \cdot 2^{k+2}$ distinct fuzzy subgroups.

The group G has 3 maximal subgroups $H_1 = \mathbb{Z}_{p^k} \times \mathbb{Z}_{q^3} \times \mathbb{Z}_r$, $H_2 = \mathbb{Z}_{p^{k+1}} \times \mathbb{Z}_{q^2} \times \mathbb{Z}_r$ and $H_3 = \mathbb{Z}_{p^{k+1}} \times \mathbb{Z}_{q^3}$ through which all maximal chains of G pass. These maximal chains are sketched as

$$p^{k+1}q^3r \supseteq p^kq^3r \supseteq \begin{cases} \cdots \\ \cdots \\ \cdots \end{cases}, \quad p^{k+1}q^3r \supseteq p^{k+1}q^2r \supseteq \begin{cases} \cdots \\ \cdots \\ \cdots \end{cases} \quad \text{and}$$

$$p^{k+1}q^3r \supseteq p^{k+1}q^3 \supseteq \begin{cases} \cdots \\ \cdots \end{cases}$$

As in Proposition 3.1–3.2, we first proceed along these three subgroups to find the number of chains with a distinguishing factor, a distinguishing pair and a distinguishing triple. Then in the fourth case, we look at the number of maximal chains with a distinguishing quadruple in both H_2 and H_3.

<u>*Case* (i):</u> $H_1 = \mathbb{Z}_{p^k} \times \mathbb{Z}_{q^3} \times \mathbb{Z}_r$

By the inductive hypothesis, H_1 has $2^{k+6} - 1 + (7k+3) \cdot 2^{k+5} + \frac{(15k^2+3k)}{2!} \cdot 2^{k+4} + \frac{(13k^2+k)(k-1)}{3!} \cdot 2^{k+3} + \frac{k^2(k-1)(k-2)}{3!} \cdot 2^{k+2}$ distinct fuzzy subgroups since maximal chains of G have length $k+6$.

<u>*Case* (ii):</u> $H_2 = \mathbb{Z}_{p^{k+1}} \times \mathbb{Z}_{q^2} \times \mathbb{Z}_r$

The subgroup H_2 has 6 maximal chains in Figure 21 with a distinguishing factor.

$$
\begin{gathered}
p^{k+1}q^3r \supseteq p^{k+1}q^2r^* \supseteq p^kq^2r \supseteq p^{k-1}q^2r \supseteq p^{k-2}q^2r \supseteq \cdots \supseteq pq^2r \supseteq pqr \supseteq pq \supseteq p \supseteq 0 \\
p^{k+1}q^3r \supseteq p^{k+1}q^2r \supseteq p^{k+1}qr^* \supseteq p^kqr \supseteq p^{k-1}qr \supseteq \cdots \supseteq p^2qr \supseteq pqr \supseteq pq \supseteq p \supseteq 0 \\
p^{k+1}q^3r \supseteq p^{k+1}q^2r \supseteq p^{k+1}qr \supseteq p^{k+1}q^* \supseteq p^kq \supseteq \cdots \supseteq p^3q \supseteq p^2q \supseteq pq \supseteq p \supseteq 0 \\
p^{k+1}q^3r \supseteq p^{k+1}q^2r \supseteq p^{k+1}qr \supseteq p^{k+1}q \supseteq p^{k+1^*} \supseteq \cdots \supseteq p^4 \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0 \\
p^{k+1}q^3r \supseteq p^{k+1}q^2r \supseteq p^{k+1}qr \supseteq p^{k+1}r \supseteq p^kr^* \supseteq \cdots \supseteq p^3r \supseteq p^2r \supseteq pr \supseteq p \supseteq 0 \\
p^{k+1}q^3r \supseteq p^{k+1}q^2r \supseteq p^{k+1}q^2 \supseteq p^kq^2 \supseteq p^{k-1}q^{2^*} \supseteq \cdots \supseteq p^2q^2 \supseteq pq^2 \supseteq pq \supseteq p \supseteq 0
\end{gathered}
$$

FIGURE 21. Maximal chains of $H_2 = p^{k+1}q^2r$ with a distinguishing factor

The subgroup H_2 has 7 clusters of maximal chains with a distinguishing pair. The process of enumerating these chains follows a similar technique as shown in the proof for Proposition 3.2. Therefore, we simply state the representative of each cluster and the number of chains therein.

The first and second clusters consist respectively of chains ending with $pq \supseteq p \supseteq 0$ and $p^2 \supseteq p \supseteq 0$ and each cluster has $(5k-3)$ maximal chains. The third cluster has 4 chains ending with $pq \supseteq q \supseteq 0$. The fourth and fifth clusters' chains end respectively with $pr \supseteq r \supseteq 0$ and $qr \supseteq q \supseteq 0$ and each has 3 chains. The sixth cluster has $2k$ chains all of which end with $pr \supseteq p \supseteq 0$; the seventh cluster has 2 chains both of which end with $q^2 \supseteq q \supseteq 0$. These 7 clusters give a total of $2(5k-3)+4+6+2 = (12k+6)$ maximal chains through H_2 with a distinguishing pair.

For maximal chains with a distinguishing triple, the subgroups p^2q^2r, p^3q^2r, p^4q^2r, $p^5q^2r, \ldots$ have 6, 22, 48, 84, … maximal chains respectively. Therefore, as k increases, the number of chains for $p^{k+1}q^2r$ with a distinguishing triple exhibits a quadratic sequence. The n^{th} term of the sequence is given by $T_n = an^2+bn+c$, where $2a =$ first term, $3a+b =$ first term of the first difference row and $2a =$ second term. This sequence is illustrated in Figure 22(A).

$$
\begin{array}{ccccccc}
6 & & 22 & & 48 & & 84 \;\cdots \\
& 16 & & 16 & & 36 & \\
& & 10 & & 10 & &
\end{array}
$$

(A) Triples for $H_2 = p^{k+1}q^2r$

$$
\begin{array}{ccccccc}
2 & & 7 & & 15 & & 26 \;\cdots \\
& 5 & & 8 & & 11 & \\
& & 3 & & 3 & &
\end{array}
$$

(B) Quadruples for $H_2 = p^{k+1}q^2r$ and $H_3 = p^{k+1}q^3$

FIGURE 22. Distinguishing triples and quadruples

We therefore have the system $2a = 10$, $3a+b = 16$ and $a+b+c = 6$, whose solution is $a = 5$, $b = 1$ and $c = 0$. For our sequence, $n = k$, implying that $T_k = 5k^2+k = k(5k+1)$ is the number of maximal chains in H_2 with a distinguishing triple.

Case (iii): $H_3 = \mathbb{Z}_{p^{k+1}} \times \mathbb{Z}_{q3}$
There is 1 maximal chain passing through H_3 with a distinguishing factor. This is the chain $p^{k+1}q^3r \supseteq p^{k+1}q^{3^*} \supseteq p^kq^3 \supseteq p^{k-1}q^3 \supseteq \cdots \supseteq p^2q^3 \supseteq pq^3 \supseteq pq^2 \supseteq pq \supseteq p \supseteq 0$.

The subgroup H_3 has 4 clusters of maximal chains with a distinguishing pair. The first cluster has all its maximal chains ending with $pq \supseteq p \supseteq 0$ and has $2k$ such chains. The second and third clusters' maximal chains end with $pq \supseteq q \supseteq 0$ and $q^2 \supseteq q \supseteq 0$ respectively and have respectively 1 and 2 maximal chain(s). The fourth cluster consists of k maximal chains all of which end with $p^2 \supseteq p \supseteq 0$. These 4 clusters contribute a total of $(3k+3)$ maximal chains with a distinguishing pair. Similarly, H_3 has $\frac{k(3k+3)}{2!}$ maximal chains with a distinguishing triple.
Case (iv): *Quadruples in both* H_2 *and* H_3
Lastly, we look at the number of chains with a distinguishing quadruple contributed by H_2 and H_3 combined. This last category has four clusters. The first cluster has $(k-1)$ maximal chains all of which end with $qr \supseteq r \supseteq 0$. The second cluster consists of chains ending with $pr \supseteq r \supseteq 0$. The number of chains with a distinguishing quadruple in this cluster for the groups p^3q^3r, p^4q^3r, p^5q^3r, p^6q^3r, $p^7q^3r, \ldots$, is 1, 3, 6, 10, $15, \ldots$, respectively. This is a sequence of triangular numbers whose n-th term is given by $T_n - \frac{n(n+1)}{2!}$. In our case, $n = k-2$. Therefore, the second cluster has $T_{k-2} = \frac{(k-1)(k-2)}{2!}$ chains.

The third cluster has chains ending with $pq \supseteq q \supseteq 0$. The number of chains in this cluster for the groups p^3q^3r, p^4q^3r, p^5q^3r, $p^6q^3r, \ldots$, is 2, 7, 15, $26, \ldots$ respectively. This is a sequence of quadratic numbers as shown in Figure 22(B). In our case, $n = k-1$ thus we have $2a = 3$, $3a + b = 5$ and $a + b + c = 2$. Solving this system gives $a = \frac{3}{2}$, $b = \frac{1}{2}$ and $c = 0$. So $T_n = \frac{3}{2}n^2 + \frac{1}{2}n = \frac{n(3n+1)}{2!}$ and therefore, the third cluster has $T_{k-1} = \frac{(k-1)[3(k-1)+1]}{2!} = \frac{(k-1)(3k-2)}{2!}$ maximal chains. The fourth cluster consists of chains $\frac{(k-1)(k-2)(4k-3)}{3!}$ all of which end with $p^2 \supseteq p \supseteq 0$. The sum from this case yields $(k-1) + \frac{(k-1)(k-2)}{2!} + \frac{(k-1)(3k-2)}{2!} + \frac{(k-1)(k-2)(4k-3)}{3!} = \frac{k(k-1)(4k+1)}{3!}$ maximal chains.

Summing up the contributions from case (i)–case (iv), we get $[2^{k+6} - 1 + (7k+3)\cdot 2^{k+5} + \frac{(15k^2+3k)}{2!}\cdot 2^{k+4} + \frac{(13k^2+k)(k-1)}{3!}\cdot 2^{k+3} + \frac{k^2(k-1)(k-2)}{3!}\cdot 2^{k+2}] + [6+1]\cdot 2^{k+5} + [(12k+6)+(3k+3)]\cdot 2^{k+4} + [k(5k+1)+\frac{k(3k+3)}{2!}]\cdot 2^{k+3} + [\frac{k(k-1)(4k+1)}{3!}]\cdot 2^{k+2}$.

Therefore G has $2^{k+6} - 1 + [7(k+1)+3]\cdot 2^{k+5} + \frac{15(k+1)^2+3(k+1)}{2!}\cdot 2^{k+4} + \frac{[13(k+1)^2+(k+1)]k}{3!}\cdot 2^{k+3} + \frac{(k+1)^2k(k-1)}{3!}\cdot 2^{k+2}$ distinct fuzzy subgroups. This can also be obtained by substituting $n = k+1$ in $2^{n+5} - 1 + (7n+3)\cdot 2^{n+4} + \frac{(15n^2+3n)}{2!}\cdot 2^{n+3} + \frac{(13n^2+n)(n-1)}{3!}\cdot 2^{n+2} + \frac{n^2(n-1)(n-2)}{3!}\cdot 2^{n+1}$. This completes the proof. $\square$

4. Conclusion

Using the criss-cut method, this paper has discussed and given in polynomial formulas the number of distinct fuzzy subgroups of $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^m} \times \mathbb{Z}_r$ for $n,\ m \in \mathbb{Z}^+$ for the cases $m = 1,\ 2,\ 3$. An immediate question in extending this work would be to get general results for all value of m in $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^m} \times \mathbb{Z}_r$. One would also extend this work to the group $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^m} \times \mathbb{Z}_{r^s}$, $n,\ m \in \mathbb{Z}^+$. This forms a basis of our next research and paper in the future.

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