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THE NUMBER OF DISTINCT FUZZY SUBGROUPS OF THE GROUP $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^m} \times \mathbb{Z}_r$ FOR DISTINCT PRIMES p, q, r AND $m, n \in \mathbb{Z}^+$

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Abstract

The equivalence relation '~' defined by Murali and Makamba is used to find the number of the distinct fuzzy subgroups of the group $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^m} \times \mathbb{Z}_r$, where p, q, r are distinct primes with m and n as

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positive integers. Using the criss-cut method explained in this paper, explicit formulae are presented.

1. Introduction

There are a lot of applications of fuzzy group theory in different fields like computer science, electrical engineering and medicine.

Zadeh introduced the notion of fuzzy sets. Reference [8] made contribution to the area of fuzzy subgroups. In this paper, we use $p^n q^m r$ to denote the group $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^m} \times \mathbb{Z}_r$, where p, q, r are distinct primes and $n, m \in \mathbb{Z}^+$. This paper is an extension of the work on the number of fuzzy subgroups of the group $p^n q^m r$ for $m = 1, 2, 3$ done by [1] using the criss-cut counting technique in [5, 7] which is anchored on the equivalence relation in [2]. Using this equivalence relation, we now get the number of fuzzy subgroups of the abelian group $p^n q^m r$ for $n, m \in \mathbb{Z}^+$.

Preliminary definitions and propositions required later in the paper are first presented.

2. Preliminaries

Definition 2.1. A *maximal chain* is a chain of subgroups of a group that cannot properly be contained in another chain.

In this paper, a chain refers to a maximal chain of subgroups.

Proposition 2.2 [6]**.** *The group* $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^m} \times \mathbb{Z}_{r^s}$ *has* $\dfrac{(n+m+s)!}{n!m!s!}$ *maximal chains*.

A set can be described by enumerating its elements using a characteristic function which is a function that assigns a value 0 and 1 to each element in the universe of discourse based on membership or non-membership.

Definition 2.3. The *characteristic function* or *indicator function* of a subset A of a set X is the function $\chi_A : X \to \{0, 1\}$ defined by

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

A fuzzy subset of a non-empty set X is a membership function $\mu : X \to I = [0, 1]$.

Definition 2.4 [2]**.** Subgroups μ and ν of G are *equivalent* denoted $\mu \sim \nu$ if

(i) for every $x, y \in X$, $\mu(x) > \mu(y)$ if and only if $\nu(x) > \nu(y)$,

(ii) $\mu(x) = 0$ if and only if $\nu(x) = 0$.

Definition 2.5 [8]**.** A fuzzy subset μ of a group G is a *fuzzy subgroup* of G, if for all $x, y \in G$:

(i) $\mu(xy) \geq \min\{m(x), \mu(y)\}$,

(ii) $\mu(x^{-1}) \geq \mu(x)$.

Ndiweni and Makamba in [7] gave a detailed explanation of the criss-cut counting technique which we summarize below.

Criss-cut counting technique

Suppose G is a finite group with all its maximal chains having length $n + 1$ and let the first chain be

$$G_0 = 0 \subseteq G_1 \subseteq G_2 \subseteq \cdots \subseteq G_n = G. \quad (1)$$

This chain contributes $2^{n+1} - 1$ distinct fuzzy subgroups to G [2]. Any other maximal chain having a subgroup H not appearing in the first chain, contributes $\frac{2^{n+1}}{2}$ distinct fuzzy subgroups. We usually call such an H, a *distinguishing factor* of the chain. If this second chain has more than one subgroup not appearing in the first chain, then we choose only one of them as a distinguishing factor.

Once all the (single) distinguishing factors have been exhausted, then we identify a pair of distinguishing factors in a chain. Such a chain contributes $\frac{2^{n+1}}{3}$ distinct fuzzy subgroups. After exhausting pairs of distinguishing factors, we identify a triple of distinguishing factors (if any). Such a chain contributes $\frac{2^{n+1}}{4}$ distinct fuzzy subgroups. The process of distinguishing factors is continued until all maximal chains get exhausted.

Summing all such contributions will yield the total number of distinct fuzzy subgroups of G.

Proposition 2.6. *If a maximal subgroup chain of length* $n + 1$, *other than the first chosen chain, has no distinguishing* $(m - 1)$*-tuple, but has a distinguishing m-tuple for* $m \geq 2$, *then the chain contributes* $\frac{2^{n+1}}{2^m} = 2^{n+1-m}$ *new distinct fuzzy subgroups of a finite group* G, $n + 1 > m$.

$\{*\}$ denotes a distinguishing factor, $\{*, **\}$ denotes a distinguishing pair and similarly other distinguishing n-tuples are denoted.

Proposition 2.7 [3]**.** *If two fuzzy subgroups are non-equivalent, then they are distinct.*

The number of distinct fuzzy subgroups of $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^m}$ is given in Theorem 2.8.

Theorem 2.8 [4]**.** *The group* $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^m}$ *has*

$$\left[2^{n+m+1}\sum_{r=0}^{m} 2^{-r}\binom{n}{r}\binom{m}{r}\right] - 1, \quad n \geq m$$

distinct fuzzy subgroups.

3. Distinct Fuzzy Subgroups of $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^m} \times \mathbb{Z}_r$, $n, m \in \mathbb{Z}^+$; p, q, r Distinct Primes

From the maximal chains, we determine the fuzzy subgroups of $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^m} \times \mathbb{Z}_r$, using the criss-cut method given in [4].

3.1. Distinct fuzzy subgroups of $\mathbb{Z}_{p^n} \times \mathbb{Z}_q \times \mathbb{Z}_r$

For $n = 1, 2$, $\mathbb{Z}_p \times \mathbb{Z}_q \times \mathbb{Z}_r$ and $\mathbb{Z}_{p^2} \times \mathbb{Z}_q \times \mathbb{Z}_r$ have, respectively, $1 \cdot (2^4 - 1) + 4 \cdot 2^3 + 1 \cdot 2^2 = 51$ and $1 \cdot (2^5 - 1) + 7 \cdot 2^4 + 4 \cdot 2^3 = 175$ distinct fuzzy subgroups (Figure 1).

$$
\begin{array}{ll}
 & p^2qr \supseteq qpr \supseteq pq \supseteq p \supseteq 0 : 2^5 - 1 \\
 & p^2qr \supseteq qpr \supseteq pq \supseteq q \supseteq 0 : 2^4 \\
 & p^2qr \supseteq qpr \supseteq pr \supseteq p \supseteq 0 : 2^4 \\
pqr \supseteq pq \supseteq p \supseteq 0 : 2^4 - 1 & p^2qr \supseteq qpr \supseteq pr \supseteq r \supseteq 0 : 2^4 \\
pqr \supseteq pq \supseteq q \supseteq 0 : 2^3 & p^2qr \supseteq qpr \supseteq qr \supseteq q \supseteq 0 : 2^4 \\
pqr \supseteq pr \supseteq p \supseteq 0 : 2^3 & p^2qr \supseteq qpr \supseteq qr \supseteq r \supseteq 0 : 2^3 \\
pqr \supseteq pr \supseteq r \supseteq 0 : 2^3 & p^2qr \supseteq p^2q \supseteq pq \supseteq p \supseteq 0 : 2^4 \\
pqr \supseteq qr \supseteq q \supseteq 0 : 2^3 & p^2qr \supseteq p^2q \supseteq pq \supseteq q \supseteq 0 : 2^3 \\
pqr \supseteq qr \supseteq r \supseteq 0 : 2^2 & p^2qr \supseteq p^2q \supseteq p^2 \supseteq p \supseteq 0 : 2^4 \\
 & p^2qr \supseteq p^2r \supseteq pr \supseteq p \supseteq 0 : 2^4 \\
 & p^2qr \supseteq p^2r \supseteq pr \supseteq r \supseteq 0 : 2^3 \\
 & p^2qr \supseteq p^2r \supseteq p^2 \supseteq p \supseteq 0 : 2^3
\end{array}
$$

Figure 1. Fuzzy subgroups of pqr and p^2qr.

For $n = 3, 4$, $\mathbb{Z}_{p^3} \times \mathbb{Z}_q \times \mathbb{Z}_r$ and $\mathbb{Z}_{p^4} \times \mathbb{Z}_q \times \mathbb{Z}_r$ have, respectively, $1 \cdot (2^6 - 1) + 10 \cdot 2^5 + 9 \cdot 2^4 = 527$ and $1 \cdot (2^7 - 1) + 13 \cdot 2^6 + 16 \cdot 2^5 = 1471$ distinct fuzzy subgroups (Figure 2 and Figure 3).

$$p^3qr \supseteq qpr \supseteq pq \supseteq p \supseteq 0 : 2^6 - 1$$
$$p^3qr \supseteq p^2qr \supseteq qpr \supseteq pq \supseteq q \supseteq 0 : 2^5$$
$$p^3qr \supseteq p^2qr \supseteq qpr \supseteq pr \supseteq p \supseteq 0 : 2^5$$
$$p^3qr \supseteq p^2qr \supseteq qpr \supseteq pr \supseteq r \supseteq 0 : 2^5$$
$$p^3qr \supseteq p^2qr \supseteq qpr \supseteq qr \supseteq q \supseteq 0 : 2^5$$
$$p^3qr \supseteq p^2qr \supseteq qpr \supseteq qr \supseteq r \supseteq 0 : 2^4$$
$$p^3qr \supseteq p^2qr \supseteq p^2q \supseteq pq \supseteq p \supseteq 0 : 2^5$$
$$p^3qr \supseteq p^2qr \supseteq p^2q \supseteq pq \supseteq q \supseteq 0 : 2^4$$
$$p^3qr \supseteq p^2qr \supseteq p^2q \supseteq p^2 \supseteq p \supseteq 0 : 2^5$$
$$p^3qr \supseteq p^2qr \supseteq p^2r \supseteq pr \supseteq p \supseteq 0 : 2^5$$

$$\underset{Ctd\cdots}{\Rightarrow}$$

$$p^3qr \supseteq p^2qr \supseteq p^2r \supseteq pr \supseteq r \supseteq 0 : 2^4$$
$$p^3qr \supseteq p^2qr \supseteq p^2r \supseteq p^2 \supseteq p \supseteq 0 : 2^4$$
$$p^3qr \supseteq p^3q \supseteq p^2q \supseteq pq \supseteq p \supseteq 0 : 2^5$$
$$p^3qr \supseteq p^3q \supseteq p^2q \supseteq pq \supseteq q \supseteq 0 : 2^4$$
$$p^3qr \supseteq p^3q \supseteq p^2q \supseteq p^2 \supseteq p \supseteq 0 : 2^4$$
$$p^3qr \supseteq p^3q \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0 : 2^5$$
$$p^3qr \supseteq p^3r \supseteq p^2r \supseteq pr \supseteq p \supseteq 0 : 2^5$$
$$p^3qr \supseteq p^3r \supseteq p^2r \supseteq pr \supseteq r \supseteq 0 : 2^4$$
$$p^3qr \supseteq p^3r \supseteq p^2r \supseteq p^2 \supseteq p \supseteq 0 : 2^4$$
$$p^3qr \supseteq p^3r \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0 : 2^4$$

Figure 2. Fuzzy subgroups of p^3qr.

$$p^4qr \supseteq p^3qr \supseteq qpr \supseteq pq \supseteq p \supseteq 0 : 2^7 - 1$$
$$p^4qr \supseteq p^3qr \supseteq p^2qr \supseteq qpr \supseteq pq \supseteq q \supseteq 0 : 2^6$$
$$p^4qr \supseteq p^3qr \supseteq p^2qr \supseteq qpr \supseteq pr \supseteq p \supseteq 0 : 2^6$$
$$p^4qr \supseteq p^3qr \supseteq p^2qr \supseteq qpr \supseteq pr \supseteq r \supseteq 0 : 2^6$$
$$p^4qr \supseteq p^3qr \supseteq p^2qr \supseteq qpr \supseteq qr \supseteq q \supseteq 0 : 2^6$$
$$p^4qr \supseteq p^3qr \supseteq p^2qr \supseteq qpr \supseteq qr \supseteq r \supseteq 0 : 2^5$$
$$p^4qr \supseteq p^3qr \supseteq p^2qr \supseteq p^2q \supseteq pq \supseteq p \supseteq 0 : 2^6$$
$$p^4qr \supseteq p^3qr \supseteq p^2qr \supseteq p^2q \supseteq pq \supseteq q \supseteq 0 : 2^5$$
$$p^4qr \supseteq p^3qr \supseteq p^2qr \supseteq p^2q \supseteq p^2 \supseteq p \supseteq 0 : 2^6$$
$$p^4qr \supseteq p^3qr \supseteq p^2qr \supseteq p^2r \supseteq pr \supseteq p \supseteq 0 : 2^6$$
$$p^4qr \supseteq p^3qr \supseteq p^2qr \supseteq p^2r \supseteq pr \supseteq r \supseteq 0 : 2^5$$
$$p^4qr \supseteq p^3qr \supseteq p^2qr \supseteq p^2r \supseteq p^2 \supseteq p \supseteq 0 : 2^5$$
$$p^4qr \supseteq p^3qr \supseteq p^3q \supseteq p^2q \supseteq pq \supseteq p \supseteq 0 : 2^6$$
$$p^4qr \supseteq p^3qr \supseteq p^3q \supseteq p^2q \supseteq pq \supseteq q \supseteq 0 : 2^5$$
$$p^4qr \supseteq p^3qr \supseteq p^3q \supseteq p^2q \supseteq p^2 \supseteq p \supseteq 0 : 2^5$$

$$\underset{Ctd\cdots}{\Rightarrow}$$

$$p^4qr \supseteq p^3qr \supseteq p^3q \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0 : 2^6$$
$$p^4qr \supseteq p^3qr \supseteq p^3r \supseteq p^2r \supseteq pr \supseteq p \supseteq 0 : 2^6$$
$$p^4qr \supseteq p^3qr \supseteq p^3r \supseteq p^2r \supseteq pr \supseteq r \supseteq 0 : 2^5$$
$$p^4qr \supseteq p^3qr \supseteq p^3r \supseteq p^2r \supseteq p^2 \supseteq p \supseteq 0 : 2^5$$
$$p^4qr \supseteq p^3qr \supseteq p^3r \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0 : 2^5$$
$$p^4qr \supseteq p^4q \supseteq p^3q \supseteq p^2q \supseteq pq \supseteq p \supseteq 0 : 2^6$$
$$p^4qr \supseteq p^4q \supseteq p^3q \supseteq p^2q \supseteq pq \supseteq q \supseteq 0 : 2^5$$
$$p^4qr \supseteq p^4q \supseteq p^3q \supseteq p^2q \supseteq p^2 \supseteq p \supseteq 0 : 2^5$$
$$p^4qr \supseteq p^4q \supseteq p^3q \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0 : 2^5$$
$$p^4qr \supseteq p^4q \supseteq p^4 \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0 : 2^6$$
$$p^4qr \supseteq p^4r \supseteq p^3r \supseteq p^2r \supseteq pq \supseteq p \supseteq 0 : 2^6$$
$$p^4qr \supseteq p^4r \supseteq p^3r \supseteq p^2r \supseteq pq \supseteq r \supseteq 0 : 2^5$$
$$p^4qr \supseteq p^4r \supseteq p^3r \supseteq p^2r \supseteq p^2 \supseteq p \supseteq 0 : 2^5$$
$$p^4qr \supseteq p^4r \supseteq p^3r \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0 : 2^5$$
$$p^4qr \supseteq p^4r \supseteq p^4 \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0 : 2^5$$

Figure 3. Fuzzy subgroups of p^4qr.

Table 1, in Appendix, gives the summary of the results and for $n > 4$ we have Proposition 3.1.

Proposition 3.1 [1]**.** *The group* $\mathbb{Z}_{p^n} \times \mathbb{Z}_q \times \mathbb{Z}_r$ *has* $2^{n+3} - 1 + (3n+1) \cdot 2^{n+2} + n^2 \cdot 2^{n+1}$ *distinct fuzzy subgroups.*

The proof to Proposition 3.1 can be found in [1].

3.2. Distinct fuzzy subgroups of $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^2} \times \mathbb{Z}_r$

The summary for the number of fuzzy subgroups of $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^2} \times \mathbb{Z}_r$ is given in Table 2 in Appendix.

From Table 2, the number of distinct fuzzy subgroups of $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^2} \times \mathbb{Z}_r$ is given by Proposition 3.2 below.

Proposition 3.2 [1]**.** *The group* $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^2} \times \mathbb{Z}_r$ *has*

$$2^{n+4} - 1 + (5n+2) \cdot 2^{n+3} + \left(\frac{7n^2 + n}{2!}\right) \cdot 2^{n+2} + \left[\frac{n^2(n-1)}{2!}\right] \cdot 2^{n+1}$$

distinct fuzzy subgroups.

Similarly, the number of fuzzy subgroups of $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^3} \times \mathbb{Z}_r$ is given by Propositions 3.3, and that of $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^4} \times \mathbb{Z}_r$ by Proposition 3.4. The proofs of Propositions 3.2 and 3.3 can also be found in [1], while that of Proposition 3.4 can be constructed using a similar approach.

Proposition 3.3 [1]**.** *The group* $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^3} \times \mathbb{Z}_r$ *has*

$$2^{n+5} - 1 + (7n+3) \cdot 2^{n+4} + \frac{(15n^2 + 3n)}{2!} \cdot 2^{n+3}$$

$$+ \frac{(13n^2 + n)(n-1)}{3!} \cdot 2^{n+2} + \frac{n^2(n-1)(n-2)}{3!} \cdot 2^{n+1}$$

distinct fuzzy subgroups.

Proposition 3.4 [1]**.** *The group* $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^4} \times \mathbb{Z}_r$ *has*

$$2^{n+6} - 1 + (9n+4) \cdot 2^{n+5} + \frac{(26n^2 + 6n)}{2!}$$

$$\cdot\, 2^{n+4} + \frac{(34n^2 + 4n)(n-1)}{3!} \cdot 2^{n+3} + \frac{(21n^2 + n)(n-1)(n-2)}{4!} \cdot 2^{n+2}$$

$$+ \frac{n^2(n-1)(n-2)(n-3)}{4!} \cdot 2^{n+1}$$

distinct fuzzy subgroups.

A similar approach can be used to get the number of distinct fuzzy subgroups for $p^n q^m r$ when $m = 5, 6, ..., 10.$ From these, the most general case, i.e., $m = k$ is deduced. These results are summarized in Table 3 in Appendix.

Remark 3.5. From Table 4, it is observed that the number of terms stabilizes for $n \geq m.$ Therefore, the group $p^n q^m$ has $(m+1)$ terms while $q^n q^m r$ has $(m+2)$ terms where $n \geq m.$

To get the general formula for counting the fuzzy subgroups of $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^m} \times \mathbb{Z}_r,$ we look at the coefficients of the powers of 2 in $p^n q^m r,$ for $m = 1, 2 ..., 10$ from Table 3. This is done by analyzing each coefficient (starting with the second one, since the first terms can be deduced independently) over the increasing values of *m*. The objective is to establish a general pattern when $m = k$ for that particular coefficient. The powers of 2 can be easily generalized by observation. We thus find a generalization for the coefficients of the powers of 2. Together with the general powers of 2, we have the counting formula.

The first terms of $p^n q^m r$ are, respectively,

$$(2^{n+3} - 1), (2^{n+4} - 1), (2^{n+5} - 1), ..., (2^{n+12} - 1)$$

for $m = 1, 2 ..., 10.$ From this, we see that for $m = k,$ the first general term is

$$(2^{n+k+2} - 1).$$

The second coefficients are

$$(3n + 1), (5n + 2), (7n + 3), (9n + 4), ..., (21n + 10).$$

The coefficients of n here are 3, 5, 7 and 9. So when $m = k$, $(2k + 1)$ is the coefficient of n. The constant term is equal to m. The powers of 2 are $2^{n+2}, 2^{n+3}, 2^{n+4}, ..., 2^{n+11}$ which, in general, is 2^{n+k+1}. Thus, the second term can, in general, be written as

$$[(2m + 1)n + m] \cdot 2^{n+m+1}.$$

For the third term, we look at the pattern emanating from two cases: when m is odd and n is even.

Case (i). *m* odd.

We have the coefficients $\frac{(2n^2)}{2}$, $\frac{3(5n^2 + n)}{2}$, $\frac{5(8n^2 + 2n)}{2}$, $\frac{7(11n^2 + 3n)}{2}$, $\frac{9(14n^2 + 4n)}{2}$ when $m = 1, 3, 5, 7, 9,$ respectively. We now find the general cases for the constants in these coefficients. The constants 1, 3, 5, 7, 9 in these terms are equal to m. The coefficients of n^2 are 2, 5, 8, 11, 14 and form an arithmetic progression with first term 2, common difference 3 and the number of terms is $\frac{m+1}{2}$. Therefore the nth term of this arithmetic sequence is

$$\begin{aligned} T_n &= 2 + 3\left(\frac{(m+1)}{2} - 1\right) \\ &= 2 + \frac{3(m+1)}{2} - 3 \\ &= \frac{3m+1}{2}. \end{aligned}$$

So when $m = k$, the coefficient of n^2 is $\frac{3k+1}{2}$.

The coefficients of n in these terms are 0, 1, 2, 3, 4 and are equal to $\frac{k-1}{2}$ for $m = k$. The powers of 2 are $2^{n+1}, 2^{n+2}, 2^{n+3}, 2^{n+4}, \ldots, 2^{n+10}$, generally yielding 2^{n+k}. Therefore, when $m = k$, the third term is generally given by

$$\frac{k\left[\left(\frac{3k+1}{2}\right)\cdot n^2 + \left(\frac{k-1}{2}\right)\cdot n\right]}{2} = \frac{k[(3k+1)\cdot n^2 + (k-1)\cdot n]}{4}.$$

Case (ii). *m* even.

We have the coefficients $\frac{1(7n^2+n)}{2}$, $\frac{2(13n^2+3n)}{2}$, $\frac{3(19n^2+5n)}{2}$, $\frac{4(25n^2+7n)}{2}$, $\frac{5(31n^2+9n)}{2}$, when $m = 2, 4, 6, 8, 10,$ respectively. The constants 1, 2, 3, 4, 5 are equal to $\frac{m}{2}$. The coefficients 7, 13, 19, 25, 31 of n^2 are in arithmetic progression whose first term is 7, common difference is 6 and $\frac{m}{2}$ is the number of terms. Thus, the *n*th term for this sequence is

$$T_n = 7 + 6\left(\frac{m}{2} - 1\right)$$

$$= 3m + 1.$$

Therefore, when $m = k$, the coefficient of n^2 is $(3k+1)$. The coefficients of n are 1, 3, 5, 7, 9, which give $(k-1)$ in the general case when $m = k$. Therefore, when $m = k$, the general third term is given by

$$\frac{\frac{k}{2}[(3k+1)\cdot n^2 + (k-1)\cdot n]}{2} = \frac{k[(3k+1)\cdot n^2 + (k-1)\cdot n]}{4}.$$

From the two cases, we have, in general, the third coefficient given by

$$\frac{m[(3m+1)\cdot n^2 + (m-1)\cdot n]}{4} = \binom{m}{1}\frac{n}{2}\frac{[(3m+1)n + (m-1)]}{2!}.$$

Similarly, we find that the coefficients of the powers of 2 in the 4th, 5th, 6th, ..., $(r+2)$nd, ..., $(m+1)$st (second last) and $(m+2)$nd (last) terms are, respectively:

$$\binom{m}{2}\frac{n(n-1)}{3}\frac{[(4m+1)n + (m-2)]}{3!},$$

$$\binom{m}{3}\frac{n(n-1)(n-2)}{4}\frac{[(5m+1)n + (m-3)]}{4!},$$

$$\binom{m}{4}\frac{n(n-1)(n-2)(n-3)}{5}\frac{[(6m+1)n + (m-4)]}{5!}, \ldots,$$

$$\binom{m}{r}\frac{n(n-1)\cdots(n-(r-1))}{(r+1)}\frac{[((r+2)m+1)n + (m-r)]}{(r+1)!}, \ldots,$$

$$\binom{m}{m-1}\frac{n(n-1)\cdots(n-(m-2))}{m}\frac{[((m+1)m+1)n+1]}{m!} \text{ and}$$

$$\frac{n(n-1)\cdots(n-(m-1))n}{m!}.$$

From these coefficients, we get the general formula for finding the number of distinct fuzzy subgroups of the group $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^m} \times \mathbb{Z}_r$. Therefore, our main result is Theorem 3.6.

Theorem 3.6. *The number of distinct fuzzy subgroups of the group* $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^m} \times \mathbb{Z}_r$ *is*

$$2^{n+m+2} - 1 + [(2m+1)n + m] \cdot 2^{n+m+1}$$

$$+ \binom{m}{1} \frac{n}{2} \frac{[(3m+1)n + (m-1)]}{2!} \cdot 2^{n+m}$$

$$+ \binom{m}{2} \frac{n(n-1)}{3} \frac{[(4m+1)n + (m-2)]}{3!} \cdot 2^{n+m-1}$$

$$+ \binom{m}{3} \frac{n(n-1)(n-2)}{4} \frac{[(5m+1)n + (m-3)]}{4!} \cdot 2^{n+m-2}$$

$$+ \binom{m}{4} \frac{n(n-1)(n-2)(n-3)}{5} \frac{[(6m+1)n + (m-4)]}{5!} \cdot 2^{n+m-3}$$

$$+ \binom{m}{5} \frac{n(n-1)\cdots(n-4)}{6} \frac{[(7m+1)n + (m-5)]}{6!} \cdot 2^{n+m-4}$$

$$+ \cdots + \binom{m}{r} \frac{n(n-1)\cdots(n-(r-1))}{(r+1)} \frac{[((r+2)m+1)n + (m-r)]}{(r+1)!} \cdot 2^{n+m-(r-1)}$$

$$+ \cdots + \binom{m}{m-1} \frac{n(n-1)\cdots(n-(m-2))}{m} \frac{[((m+1)m+1)n + 1]}{m!} \cdot 2^2$$

$$+ \frac{n(n-1)\cdots(n-(m-1))n}{m!} \cdot 2^{n+1}$$

$$= 2^{n+m+2} - 1 + [(2m+1)n + m] \cdot 2^{n+m+1}$$

$$+ \sum_{k=1}^{m} \binom{m}{k} \frac{n(n-1)\cdots(n-(k-1))}{(k+1)} \frac{[((k+2)m+1)n + (m-k)]}{(k+1)!}$$

$$\cdot 2^{n+m-(k-1)}, \quad 1 \le m \le n.$$

Proof. We prove the theorem by induction on n. When $n = 1$, $\mathbb{Z}_p \times \mathbb{Z}_{q^m} \times \mathbb{Z}_r \cong \mathbb{Z}_{p^m} \times \mathbb{Z}_q \times \mathbb{Z}_r$ has $2^{m+3} - 1 + (3m+1) \cdot 2^{m+2} + \frac{m \cdot m}{1!} \cdot 2^{m+1}$ distinct fuzzy subgroups, by Proposition 3.1, when $n = m$. This result can also be obtained by substituting $m = 1$ in Theorem 3.6.

Assume that the result is true for $n = k$, i.e., $\mathbb{Z}_{p^k} \times \mathbb{Z}_{q^m} \times \mathbb{Z}_r$ has

$$2^{k+m+2} - 1 + [(2m+1)k + m] \cdot 2^{k+m+1}$$

$$+ \binom{m}{1} \frac{k}{2} \frac{[(3m+1)k + (m-1)]}{2!} \cdot 2^{k+m}$$

$$+ \binom{m}{2} \frac{k(k-1)}{3} \frac{[(4m+1)k + (m-2)]}{3!} \cdot 2^{k+m-1}$$

$$+ \binom{m}{3} \frac{k(k-1)(k-2)}{4} \frac{[(5m+1)k + (m-3)]}{4!} \cdot 2^{k+m-2}$$

$$+ \binom{m}{4} \frac{k(k-1)(k-2)(k-3)}{5} \frac{[(6m+1)k + (m-4)]}{5!} \cdot 2^{k+m-3}$$

$$+ \binom{m}{5} \frac{k(k-1)\cdots(k-4)}{6} \frac{[(7m+1)k + (m-5)]}{6!} \cdot 2^{k+m-4}$$

$$+ \cdots + \binom{m}{r} \frac{k(k-1)\cdots(k-(r-1))}{(r+1)} \frac{[((r+2)m+1)k + (m-r)]}{(r+1)!} \cdot 2^{k+m-(r-1)}$$

$$+ \cdots + \binom{m}{m-1} \frac{k(k-1)\cdots(k-(m-2))}{m} \frac{[((m+1)m+1)k + 1]}{m!} \cdot 2^{k+2}$$

$$+ \frac{k(k-1)\cdots(k-(m-1))n}{m!} \cdot 2^{k+1}$$

distinct fuzzy subgroups.

We need to show that $G = \mathbb{Z}_{p^{k+1}} \times \mathbb{Z}_{q^m} \times \mathbb{Z}_r$ has

$$2^{k+m+3} - 1 + [(2m+1)(k+1) + m] \cdot 2^{k+m+2}$$

$$+ \binom{m}{1} \frac{(k+1)}{2} \frac{[(3m+1)(k+1) + (m-1)]}{2!} \cdot 2^{k+m+1}$$

$$+\binom{m}{2}\frac{(k+1)k}{3}\frac{[(4m+1)(k+1)+(m-2)]}{3!}\cdot 2^{k+m}$$

$$+\binom{m}{3}\frac{(k+1)k(k-1)}{4}\frac{[(5m+1)(k+1)+(m-3)]}{4!}\cdot 2^{k+m-1}$$

$$+\binom{m}{4}\frac{k(k+1)(k-1)(k-2)}{5}\frac{[(6m+1)(k+1)+(m-4)]}{5!}\cdot 2^{k+m-2}$$

$$+\binom{m}{5}\frac{k(k+1)(k-1)(k-2)(k-3)}{6}\frac{[(7m+1)(k+1)+(m-5)]}{6!}\cdot 2^{k+m-3}$$

$$+\cdots+\binom{m}{r}\frac{(k+1)k(k-1)\cdots(k-(r-2))}{(r+1)}$$

$$\cdot\frac{[((r+2)m+1)(k+1)+(m-r)]}{(r+1)!}\cdot 2^{k+m-(r-2)}$$

$$+\cdots+\binom{m}{m-1}\frac{(k+1)k(k-1)\cdots(k-(m-3))}{m}\frac{[((m+1)m+1)(k+1)+1]}{m!}\cdot 2^{k+3}$$

$$+\frac{(k+1)k(k-1)\cdots(k-(m-2))(k+1)}{m!}\cdot 2^{k+2}$$

distinct fuzzy subgroups.

The group G has 3 maximal subgroups $H_1 = p^k q^m r$, $H_2 = p^{k+1}q^{m-1}r$ and $H_3 = p^{k+1}q^m$. The maximal chains of G are illustrated below:

$$p^{k+1}q^m r \supseteq p^k q^m r \supseteq \begin{cases}\cdots\\ \cdots,\\ \cdots\end{cases} \quad p^{k+1}q^m r \supseteq p^{k+1}q^{m-1} r \supseteq \begin{cases}\cdots\\ \cdots\\ \cdots\end{cases} \text{ and}$$

$$p^{k+1}q^m r \supseteq p^{k+1}q^m \supseteq \begin{cases}\cdots\\ \cdots\end{cases}.$$

Case (i). $H_1 = \mathbb{Z}_{p^k} \times \mathbb{Z}_{q^m} \times \mathbb{Z}_r$

By the inductive hypothesis, the subgroup H_1 has

$$2^{k+m+3} - 1 + [(2m+1)k + m] \cdot 2^{k+m+2}$$

$$+ \binom{m}{1} \frac{k}{2} \frac{[(3m+1)k + (m-1)]}{2!} \cdot 2^{k+m+1}$$

$$+ \binom{m}{2} \frac{k(k-1)}{3} \frac{[(4m+1)k + (m-2)]}{3!} \cdot 2^{k+m}$$

$$+ \binom{m}{3} \frac{k(k-1)(k-2)}{4} \frac{[(5m+1)k + (m-3)]}{4!} \cdot 2^{k+m-1}$$

$$+ \binom{m}{4} \frac{k(k-1)(k-2)(k-3)}{5} \frac{[(6m+1)k + (m-4)]}{5!} \cdot 2^{k+m-2}$$

$$+ \binom{m}{5} \frac{k(k-1)\cdots(k-4)}{6} \frac{[(7m+1)k + (m-5)]}{6!} \cdot 2^{k+m-3}$$

$$+ \cdots + \binom{m}{r} \frac{k(k-1)\cdots(k-(r-1))}{(r+1)} \frac{[((r+2)m+1)k + (m-r)]}{(r+1)!} \cdot 2^{k+m-(r-2)}$$

$$+ \cdots + \binom{m}{m-1} \frac{k(k-1)\cdots(k-(m-2))}{m} \frac{[((m+1)m+1)k + 1]}{m!} \cdot 2^{k+3}$$

$$+ \frac{k(k-1)\cdots(k-(m-1))k}{m!} \cdot 2^{k+2}$$

distinct fuzzy subgroups (because maximal chains of G are of length $k + m + 3$).

Case (ii). Contribution from H_2 and H_3

There are $2m$ maximal chains with a distinguishing factor from H_2 and 1 maximal chain with a distinguishing factor from H_3. This gives a sum of $(2m+1)$ maximal chains with a distinguishing factor. From Case (i) and

Case (ii), we have a total of

$$[(2m+1)k+m]+(2m+1)=[(2m+1)(k+1)+m]$$

maximal chains with a distinguishing factor. This is clearly the coefficient of 2^{k+m+2} in the second term in the formula for the number of distinct fuzzy subgroups of the group G.

When $m = 1, 2, 3, 4, 5, 6,$ we have, respectively, k, $5k+2$, $12k+6$, $22k+12$, $35k+20$ and $51k+30$ maximal chains with a distinguishing pair through the maximal subgroup H_2. This pattern yields $\frac{m(3m-1)k}{2!}$ $+ m(m-1)$ maximal chains along H_2 with a distinguishing pair. Along the subgroup H_3, for $m = 1, 2, 3, 4, 5, 6,$ we have, respectively, $k+1$, $2k+2$, $3k+3$, $4k+4$, $5k+5$ and $6k+6$ maximal chains which have a distinguishing pair. Therefore, H_3 has $(mk+m)$ maximal chains which have a distinguishing pair. From H_2 and H_3, we have

$$\frac{3m^2-mk}{2!}+(m^2-m)+(mk+m)=\frac{3m^2k+mk+2m^2}{2!}$$

$$=\frac{mk(3m+1)+2m^2}{2!}$$

$$=\frac{m[(3m+1)k+2m]}{2!}$$

$$=\binom{m}{1}\frac{[(3m+1)k+2m]}{2!}.$$

Therefore, Case (ii) yields $\binom{m}{1}\frac{[(3m+1)k+2m]}{2!}$ maximal chains having a distinguishing pair.

Case (i) has $\binom{m}{1}\frac{k}{2}\frac{[(3m+1)k+(m-1)]}{2!}=\frac{m^2k^2+mk^2+m^2k-mk}{4}$ maximal chains which have a distinguishing pair. The sum of the totals from

Case (i) and Case (ii) gives

$$\frac{m^2k^2 + mk^2 + m^2k - mk}{4} + \frac{3m^2k + mk + 2m^2}{2!}$$

$$= \frac{3m^2k^2 + 7m^2k + mk^2 + mk + 4m^2}{4}$$

$$= \binom{m}{1}\frac{(k+1)}{2}\frac{[(3m+1)(k+1)+(m-1)]}{2!}.$$

Therefore, G has $\binom{m}{1}\frac{(k+1)}{2}\frac{[(3m+1)(k+1)+(m-1)]}{2!}$ maximal chains which have a distinguishing pair. This is the coefficient of 2^{k+m+1} in the third term of the formula for the number of distinct fuzzy subgroups of G.

Similarly, when $m = 1, 2, 3, 4, 5, 6,$ the subgroup H_2 has, respectively, $0,\ k^2,\ 5k(k+1),\ 2k(7k+2),\ 5k(6k+2),\ 5k(13k+4)$ maximal chains which have a distinguishing triple. From this pattern, the number of maximal chains which have a distinguishing triple through H_2 is given by

$$\frac{2m^3k^2 - 3m^2k^2 + m^3k - 3m^2k + mk^2 + 2mk}{3!}$$

$$= \frac{mk\cdot(2m^2k - 3mk + m^3 - 3m + k + 2)}{3!}$$

$$= \frac{mk\cdot[(2m^2 - 3m + 1)k + (m^2 - 3m + 2)]}{3!}$$

$$= \frac{mk\cdot[(2m-1)(m-1)k + (m-2)(m-1)]}{3!}$$

$$= \frac{m(m-1)k}{2!}\cdot\frac{[(2m-1)k + (m-2)]}{3}.$$

Moreover, the subgroup H_3 has 0, $\frac{k(k+1)}{2!}$, $\frac{3k(k+1)}{2!}$, $\frac{6k(k+1)}{2!}$, $\frac{10k(k+1)}{2!}$, $\frac{15k(k+1)}{2!}$ maximal chains with a distinguishing triple when $m = 1, 2, 3, 4, 5, 6,$ respectively. Thus, in general, H_3 has $\frac{m(m-1)}{2!} \cdot \frac{k(k+1)}{2!}$ maximal chains with a distinguishing triple. Therefore, Case (ii) contributes

$$\frac{m(m-1)k}{2!} \frac{[(2m-1)k + (m-2)]}{3} + \frac{m(m-1)}{2!} \frac{k(k+1)}{2!}$$

$$= \frac{m(m-1)k(4mk + k + 2m - 1)}{12}$$

$$= \binom{m}{2} \frac{k[(4m+1)k + (2m-1)]}{3!}$$

maximal chains with a distinguishing triple. Case (i) yields $\binom{m}{2} \frac{k(k-1)}{3} \frac{[(4m+1)k + (m-2)]}{3!}$ maximal chains with a distinguishing triple. Thus, Case (i) and Case (ii) yield

$$\binom{m}{2} \frac{k(k-1)}{3} \frac{[(4m+1)k + (m-2)]}{3!} + \binom{m}{2} \frac{k[(4m+1)k + (2m-1)]}{3!}$$

$$= \binom{m}{2} \frac{k(k+1)}{3} \frac{[(4m+1)(k+1) + (m-2)]}{3!}$$

maximal chains with a distinguishing triple. This is the coefficient of 2^{k+m} in the fourth term of the formula for the number of distinct fuzzy subgroups of G.

By a similar approach, H_2 and H_3 (Case (ii)) contribute $\binom{m}{3} \frac{k(k-1)[(5m+1)k + (2m-2)]}{4!}$ maximal chains with a distinguishing quadruple. Case (i) yields $\binom{m}{3}\frac{k(k-1)(k-2)}{4}\frac{[(5m+1)k+(m-3)]}{4!}$ maximal chains with a distinguishing quadruple. Therefore, Case (i) and Case (ii) contribute

$$\binom{m}{3}\frac{k(k-1)(k-2)}{4}\frac{[(5m+1)k+(m-3)]}{4!}$$

$$+\binom{m}{3}\frac{k(k-1)[(5m+1)k+(2m-2)]}{4!}$$

$$=\binom{m}{3}\frac{k(k+1)(k-1)}{4}\frac{[(5m+1)(k+1)+(m-3)]}{4!}$$

maximal chains with a distinguishing quadruple. This again is the coefficient of 2^{k+m-1} in the fourth term of the formula for the number of distinct fuzzy subgroups of G.

Extension of the argument discussed above gives the number of maximal chains which have a distinguishing 5-tuple, 6-tuple, ..., $(r+1)$-tuple for $r \leq m$, ..., m-tuple (second last coefficient) and $(m+1)$-tuple (last coefficient) contributed by Case (ii). From this argument, their number of maximal chains are, respectively,

$$\binom{m}{4}\frac{k(k-1)(k-2)[(6m+1)k+(2m-3)]}{5!},$$

$$\binom{m}{5}\frac{k(k-1)(k-2)(k-3)[(7m+1)k+(2m-4)]}{6!}$$

$$, ..., \binom{m}{r}\frac{k(k-1)\cdots(k-(r-2))[((r+2)m+1)k+(2m-(r-1))]}{(r+1)!}$$

$$, ..., \binom{m}{m-1}\frac{k(k-1)\cdots(k-(m-3))\{[((m+1)\,m+1)]k+[2m-((m-1)-1)]\}}{m!}$$

and $\frac{k(k-1)\cdots(k-(m-2))[(m+1)k+1]}{m!}$.

The sum of contributions by Case (i) and Case (ii) to maximal chains with a distinguishing 5-tuple, 6-tuple, ..., $(r+1)$-tuple for $r \leq m$, ..., m-tuple (second last coefficient) and $(m+1)$-tuple (last coefficient) comes to be, respectively,

$$\binom{m}{4} \frac{k(k+1)(k-1)(k-2)}{5} \frac{[(6m+1)(k+1)+(m-4)]}{5!},$$

$$\binom{m}{5} \frac{k(k+1)(k-1)(k-2)(k-3)}{6} \frac{[(7m+1)(k+1)+(m-5)]}{6!}$$

$$, ..., \binom{m}{r} \frac{(k+1)k(k-1)\cdots(k-(r-2))}{(r+1)} \frac{[((r+2)m+1)(k+1)+(m-r)]}{(r+1)!}$$

$$, ..., \binom{m}{m-1} \frac{(k+1)k(k-1)\cdots(k-(m-3))}{m} \frac{\{[(m+1)m+1](k+1)+1\}}{m!}$$

$$\text{and } \frac{(k+1)k(k-1)\cdots(k-(m-2))(k+1)}{m!}.$$

These terms are the coefficient of $2^{k+m-2}, 2^{k+m-3}, ..., 2^{k+m-(r-2)}$, ..., 2^{k+3} and 2^{k+2}, respectively, in the 6th, 7th, ..., $(r+2)$nd, ..., $(m+1)$st and the $(m+2)$nd (last) terms of the formula for the number of distinct fuzzy subgroups of G.

In particular, we use the $(r+1)$-tuple and $(m+1)$-tuple to demonstrate the sum from Case (i) and Case (ii). We label the number of maximal chains with $(r+1)$-tuple from Case (i) and Case (ii) by the expressions (2) and (3), respectively,

$$\binom{m}{r} \frac{k(k-1)\cdots(k-(r-1))}{(r+1)} \frac{[((r+2)m+1)k+(m-r)]}{(r+1)!}, \tag{2}$$

$$\binom{m}{r} \frac{k(k-1)\cdots(k-(r-2))[((r+2)m+1)k+(2m-(r-1))]}{(r+1)!}. \tag{3}$$

Suppose that the sum of expressions (2) and (3) is z. Then z can be simplified as follows:

$$z = \binom{m}{r} \frac{k(k-1)\cdots(k-(r-2))(k+1)(rmk + rm + k + 2mk + 3m - r + 1)}{(r+1)(r+1)!}$$

$$= \binom{m}{r} \frac{(k+1)k(k-1)\cdots(k-(r-2))}{(r+1)}$$

$$\cdot \frac{[(rmk + 2mk + k + rm + 2m + 1) + (m - r)]}{(r+1)!}$$

$$= \binom{m}{r} \frac{(k+1)k(k-1)\cdots(k-(r-2))}{(r+1)} \frac{[(rm + 2m + 1)(k+1) + (m - r)]}{(r+1)!}$$

$$= \binom{m}{r} \frac{(k+1)k(k-1)\cdots(k-(r-2))}{(r+1)} \frac{[((r+2)m + 1)(k+1) + (m - r)]}{(r+1)!}.$$

When $r = m$ in z, we get the $(m+1)$-tuple, the last coefficient as discussed next:

$$z = \binom{m}{m} \frac{(k+1)k(k-1)\cdots(k-(m-2))}{(m+1)} \frac{[((m+2)m + 1)(k+1) + (m - m)]}{(m+1)!}$$

$$= \frac{(k+1)k(k-1)\cdots(k-(m-2))}{(m+1)} \frac{(m^2 + 2m + 1)(k+1)}{(m+1)!}$$

$$= \frac{(k+1)k(k-1)\cdots(k-(m-2))}{(m+1)} \frac{(m+1)(m+1)(k+1)}{(m+1)!}$$

$$= \frac{(k+1)k(k-1)\cdots(k-(m-2))(k+1)}{m!}.$$

The summation for the other tuples can be done similarly. We can achieve this result by letting $n = k + 1$ in Theorem 3.6 completing the proof.

□

4. Conclusion

Using the criss-cut method, the number of distinct fuzzy subgroups of $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^m} \times \mathbb{Z}_r$ for $n, m \in \mathbb{Z}^+$ has been discussed in this paper. An extension of this work would be to the group $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^m} \times \mathbb{Z}_{r^s}$, for any distinct prime numbers p, q and r and for positive integral values $s \leq m \leq n$.

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Appendix

Table 1. Fuzzy subgroups of $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^m} \times \mathbb{Z}_r$

n	$p^n qr$	Number of fuzzy subgroups
1	pqr	$51 = 1 \cdot (2^4 - 1) + 4 \cdot 2^3 + 1 \cdot 2^2$
2	$p^2 qr$	$175 = 1 \cdot (2^5 - 1) + 7 \cdot 2^4 + 4 \cdot 2^3$
3	$p^3 qr$	$527 = 1 \cdot (2^6 - 1) + 10 \cdot 2^5 + 9 \cdot 2^4$
4	$p^4 qr$	$1471 = 1 \cdot (2^7 - 1) + 13 \cdot 2^6 + 16 \cdot 2^5$
5	$p^5 qr$	$3903 = 1 \cdot (2^8 - 1) + 16 \cdot 2^7 + 25 \cdot 2^6$
6	$p^6 qr$	$9983 = 1 \cdot (2^9 - 1) + 19 \cdot 2^8 + 36 \cdot 2^7$
7	$p^7 qr$	$24831 = 1 \cdot (2^{10} - 1) + 22 \cdot 2^9 + 49 \cdot 2^8$
$\vdots$	$\vdots$	$\vdots$
K	$p^k qr$	$2^{k+3} - 1 + (3k + 1) \cdot 2^{k+2} + k^2 \cdot 2^{k+1}$

Table 2. Fuzzy subgroups of $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^2} \times \mathbb{Z}_r$

n	$p^n q^2 r$	Number of fuzzy subgroups
1	pq^2r	$175 = 1 \cdot (2^5 - 1) + 7 \cdot 2^4 + 4 \cdot 2^3$
2	p^2q^2r	$703 = 1 \cdot (2^6 - 1) + 12 \cdot 2^5 + 15 \cdot 2^4 + 2 \cdot 2^3$
3	p^3q^2r	$2415 = 1 \cdot (2^7 - 1) + 17 \cdot 2^6 + 33 \cdot 2^5 + 9 \cdot 2^4$
4	p^4q^2r	$7551 = 1 \cdot (2^8 - 1) + 22 \cdot 2^7 + 58 \cdot 2^6 + 24 \cdot 2^5$
5	p^5q^2r	$22143 = 1 \cdot (2^9 - 1) + 27 \cdot 2^8 + 90 \cdot 2^7 + 50 \cdot 2^6$
6	p^6q^2r	$61951 = 1 \cdot (2^{10} - 1) + 32 \cdot 2^9 + 129 \cdot 2^8 + 90 \cdot 2^7$
7	p^7q^2r	$167167 = 1 \cdot (2^{11} - 1) + 37 \cdot 2^{10} + 175 \cdot 2^9 + 147 \cdot 2^8$
⋮	⋮	⋮
k	p^kq^2r	$2^{k+4} - 1 + (2+5k) \cdot 2^{k+3} + \left(\frac{7k^2+k}{2}\right) \cdot 2^{k+2} + \left[\frac{k^2(k-1)}{2}\right] \cdot 2^{k+1}$

Table 3. Fuzzy subgroups of $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^m} \times \mathbb{Z}_r$

m	Group	Number of fuzzy subgroups
1	p^nqr	$2^{n+3} - 1 + (3n+1) \cdot 2^{n+2} + \frac{n \cdot n}{1!} \cdot 2^{n+1}$
2	p^nq^2r	$2^{n+4} - 1 + (5n+2) \cdot 2^{n+3}$ $+ 2 \cdot \frac{n}{2}\left(\frac{7n+1}{2!}\right) \cdot 2^{n+2} + \frac{n(n-1)n}{2!} \cdot 2^{n+1}$
3	p^nq^3r	$2^{n+5} - 1 + (7n+3) \cdot 2^{n+4} + 3 \cdot \frac{n}{2}\frac{(10n+2)}{2!} \cdot 2^{n+3}$ $+ \frac{3 \cdot 2}{2!} \cdot \frac{n(n-1)}{3}\frac{(13n+1)}{3!} \cdot 2^{n+2} + \frac{n(n-1)(n-2)n}{3!} \cdot 2^{n+1}$
4	p^nq^4r	$2^{n+6} - 1 + (9n+4) \cdot 2^{n+5} + 4 \cdot \frac{n}{2}\frac{(13n+3)}{2!} \cdot 2^{n+4}$ $+ \frac{4 \cdot 3}{2!}\frac{n(n-1)}{3}\frac{(17n+2)}{3!} \cdot 2^{n+3}$ $+ \frac{4 \cdot 3 \cdot 2}{3!} \cdot \frac{n(n-1)(n-2)}{4}\frac{(21n+1)}{4!} \cdot 2^{n+2}$ $+ \frac{n(n-1)(n-2)(n-3)n}{4!} \cdot 2^{n+1}$

5	$p^n q^5 r$	$2^{n+7} - 1 + (11n+5) \cdot 2^{n+6} + 5 \cdot \frac{n}{2} \frac{(16n+4)}{2!} \cdot 2^{n+5}$ $+ \frac{5 \cdot 4}{2!} \cdot \frac{n(n-1)}{3} \frac{(21n+3)}{3!} \cdot 2^{n+4}$ $+ \frac{5 \cdot 4 \cdot 3}{3!} \cdot \frac{n(n-1)(n-2)}{4} \frac{(26n+2)}{4!} \cdot 2^{n+3}$ $+ \frac{5 \cdot 4 \cdot 3 \cdot 2}{4!} \cdot \frac{n(n-1)(n-2)(n-3)}{5} \frac{(31n+1)}{5!} \cdot 2^{n+2}$ $+ \frac{n(n-1)(n-2)(n-3)(n-4)n}{5!} \cdot 2^{n+1}$
6	$p^n q^6 r$	$2^{n+8} - 1 + (13n+6) \cdot 2^{n+7} + 6 \cdot \frac{n}{2} \frac{(19n+5)}{2!} \cdot 2^{n+6}$ $+ \frac{6 \cdot 5}{2!} \cdot \frac{n(n-1)}{3} \frac{(25n+4)}{3!} \cdot 2^{n+5}$ $+ \frac{6 \cdot 5 \cdot 4}{3!} \cdot \frac{n(n-1)(n-2)}{4} \frac{(31n+3)}{4!} \cdot 2^{n+4}$ $+ \frac{6 \cdot 5 \cdot 4 \cdot 3}{4!} \cdot \frac{n(n-1)(n-2)(n-3)}{5} \frac{(37n+2)}{5!} \cdot 2^{n+3}$ $+ \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2}{5!} \cdot \frac{n(n-1)(n-2)(n-3)(n-4)}{6} \frac{(43n+1)}{6!} \cdot 2^{n+3}$ $+ \frac{n(n-1)(n-2)\cdots(n-5)n}{6!} \cdot 2^{n+1}$
7	$p^n q^7 r$	$2^{n+9} - 1 + (15n+7) \cdot 2^{n+8} + 7 \cdot \frac{n}{2} \frac{(22n+6)}{2!} \cdot 2^{n+7}$ $+ \frac{7 \cdot 6}{2!} \cdot \frac{n(n-1)}{3} \frac{(29n+5)}{3!} \cdot 2^{n+6}$ $+ \frac{7 \cdot 6 \cdot 5}{3!} \cdot \frac{n(n-1)(n-2)}{4} \frac{(36n+4)}{4!} \cdot 2^{n+5}$ $+ \frac{7 \cdot 6 \cdot 5 \cdot 4}{4!} \cdot \frac{n(n-1)(n-2)(n-3)}{5} \frac{(43n+3)}{5!} \cdot 2^{n+4}$ $+ \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3}{5!} \cdot \frac{n(n-1)(n-2)(n-3)(n-4)}{6} \frac{(50n+2)}{6!} \cdot 2^{n+3}$ $+ \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2}{6!} \cdot \frac{n(n-1)(n-2)\cdots(n-5)}{7} \frac{(57n+1)}{7!} \cdot 2^{n+2}$ $+ \frac{n(n-1)(n-2)\cdots(n-6)n}{7!} \cdot 2^{n+1}$

8	$p^n q^8 r$	$2^{n+10} - 1 + (17n+8)\cdot 2^{n+9} + 8 \cdot \frac{n}{2}\frac{(25n+7)}{2!}\cdot 2^{n+8}$ $+ \frac{8\cdot 7}{2!}\cdot\frac{n(n-1)}{3}\frac{(33n+6)}{3!}\cdot 2^{n+7}$ $+ \frac{8\cdot 7\cdot 6}{3!}\cdot\frac{n(n-1)(n-2)}{4}\frac{(41n+5)}{4!}\cdot 2^{n+6}$ $+ \frac{8\cdot 7\cdot 6\cdot 5}{4!}\cdot\frac{n(n-1)(n-2)(n-3)}{5}\frac{(49n+4)}{5!}\cdot 2^{n+5}$ $+ \frac{8\cdot 7\cdot 6\cdot 5\cdot 4}{5!}\cdot\frac{n(n-1)(n-2)(n-3)(n-4)}{6}\frac{(57n+3)}{6!}\cdot 2^{n+3}$ $+ \frac{8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3}{6!}\cdot\frac{n(n-1)(n-2)\cdots(n-5)}{7}\frac{(65n+2)}{7!}\cdot 2^{n+2}$ $+ \frac{8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2}{7!}\cdot\frac{n(n-1)(n-2)\cdots(n-6)}{8}\frac{(73n+1)}{8!}\cdot 2^{n+2}$ $+ \frac{n(n-1)(n-2)\cdots(n-7)n}{8!}\cdot 2^{n+1}$
9	$p^n q^9 r$	$2^{n+11} - 1 + (19n+9)\cdot 2^{n+10} + 9 \cdot \frac{n}{2}\frac{(28n+8)}{2!}\cdot 2^{n+9}$ $+ \frac{9\cdot 8}{2!}\cdot\frac{n(n-1)}{3}\frac{(37n+7)}{3!}\cdot 2^{n+8}$ $+ \frac{9\cdot 8\cdot 7}{3!}\cdot\frac{n(n-1)(n-2)}{4}\frac{(46n+6)}{4!}\cdot 2^{n+7}$ $+ \frac{9\cdot 8\cdot 7\cdot 6}{4!}\cdot\frac{n(n-1)(n-2)(n-3)}{5}\frac{(55n+5)}{5!}\cdot 2^{n+6}$ $+ \frac{9\cdot 8\cdot 7\cdot 6\cdot 5}{5!}\cdot\frac{n(n-1)(n-2)(n-3)(n-4)}{6}\frac{(64n+4)}{6!}\cdot 2^{n+5}$ $+ \frac{9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4}{6!}\cdot\frac{n(n-1)(n-2)\cdots(n-5)}{7}\frac{(73n+3)}{7!}\cdot 2^{n+4}$ $+ \frac{9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3}{7!}\cdot\frac{n(n-1)(n-2)\cdots(n-6)}{8}\frac{(82n+2)}{8!}\cdot 2^{n+3}$ $+ \frac{9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2}{8!}\cdot\frac{n(n-1)(n-2)\cdots(n-7)}{8}\frac{(91n+1)}{9!}\cdot 2^{n+2}$ $+ \frac{n(n-1)(n-2)\cdots(n-8)n}{9!}\cdot 2^{n+1}$

10	$p^n q^{10} r$	$2^{n+12} - 1 + (21n + 10) \cdot 2^{n+11} + 10 \cdot \frac{n}{2} \frac{(31n+9)}{2!} \cdot 2^{n+10}$ $+ \frac{10 \cdot 9}{2!} \cdot \frac{n(n-1)}{3} \frac{(41n+8)}{3!} \cdot 2^{n+9}$ $+ \frac{10 \cdot 9 \cdot 8}{3!} \cdot \frac{n(n-1)(n-2)}{4} \frac{(51n+7)}{4!} \cdot 2^{n+8}$ $+ \frac{10 \cdot 9 \cdot 8 \cdot 7}{4!} \cdot \frac{n(n-1)(n-2)(n-3)}{5} \frac{(61n+6)}{5!} \cdot 2^{n+7}$ $+ \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6}{5!} \cdot \frac{n(n-1)(n-2)(n-3)(n-4)}{6} \frac{(71n+5)}{6!} \cdot 2^{n+6}$ $+ \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5}{6!} \cdot \frac{n(n-1)(n-2)\cdots(n-5)}{7} \frac{(81n+4)}{7!} \cdot 2^{n+5}$ $+ \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4}{7!} \cdot \frac{n(n-1)(n-2)\cdots(n-6)}{8} \frac{(91n+3)}{8!} \cdot 2^{n+4}$ $+ \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3}{8!} \cdot \frac{n(n-1)(n-2)\cdots(n-7)}{9} \frac{(101n+2)}{9!} \cdot 2^{n+3}$ $+ \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2}{9!} \cdot \frac{n(n-1)(n-2)\cdots(n-7)}{10} \frac{(111n+2)}{10!} \cdot 2^{n+2}$ $+ \frac{n(n-1)(n-2)\cdots(n-8)n}{10!} \cdot 2^{n+1}$
$\vdots$	$\vdots$	$\vdots$
k	$p^n q^k r$	$2^{n+k+2} - 1 + [(2k+1)n + k] \cdot 2^{n+k+1}$ $+ \binom{k}{1} \frac{n}{2} \frac{[(3k+1)n + (k-1)]}{2!} \cdot 2^{n+k}$ $+ \binom{k}{2} \frac{n(n-1)}{3} \frac{[(4k+1)n + (k-2)]}{3!} \cdot 2^{n+k-1}$ $+ \binom{k}{3} \frac{n(n-1)(n-2)}{4} \frac{[(5k+1)n + (k-3)]}{4!} \cdot 2^{n+k-2}$ $+ \binom{k}{4} \frac{n(n-1)(n-2)(n-3)}{5} \frac{[(6k+1)n + (k-4)]}{5!} \cdot 2^{n+k-3}$ $+ \binom{k}{5} \frac{n(n-1)\cdots(n-4)}{6} \frac{[(7k+1)n + (k-5)]}{6!} \cdot 2^{n+k-4}$ $+ \binom{k}{6} \frac{n(n-1)\cdots(n-5)}{7} \frac{[(8k+1)n + (k-6)]}{7!} \cdot 2^{n+k-5}$

		$+\binom{k}{7}\dfrac{n(n-1)\cdots(n-6)}{8}\dfrac{[(9k+1)n+(k-7)]}{8!}\cdot 2^{n+k-6}$
		$+\binom{k}{8}\dfrac{n(n-1)\cdots(n-7)}{9}\dfrac{[(10k+1)n+(k-8)]}{9!}\cdot 2^{n+k-7}$
		$+\cdots+\binom{k}{r}\dfrac{n(n-1)\cdots(n-(r-1))}{(r+1)}\dfrac{[((r+2)k+1)n+(k-r)]}{(r+1)!}\cdot 2^{n+k-(r-1)}$
		$+\cdots+\binom{k}{k-1}\dfrac{n(n-1)\cdots[n-(k-2)]}{9}\dfrac{[((k+1)k+1)n+1]}{k!}\cdot 2^{2}$
		$+\dfrac{n(n-1)\cdots(n-(k-1))n}{k!}\cdot 2^{n+1},\ r\le k\le n$

Table 4. Number of terms in the formulae for fuzzy subgroups of p^n, p^nq, p^nq^m, p^nq^mr for $m = 1, 2, ..., 10$

n	p^n	p^nq	p^nq^2	p^nq^3	p^nqr	p^nq^2r	p^nq^3r	p^nq^4r	p^nq^5r	p^nq^6r	p^nq^7r	p^nq^8r	p^nq^9r	$p^nq^{10}r$
1	1	2	2	2	3	3	3	3	3	3	3	3	3	3
2	1	2	3	3	3	4	4	4	4	4	4	4	4	4
3	1	2	3	4	3	4	5	5	5	5	5	5	5	5
4	1	2	3	4	3	4	5	6	6	6	6	6	6	6
5	1	2	3	4	3	4	5	6	7	7	7	7	7	7
6	1	2	3	4	3	4	5	6	7	8	8	8	8	8
7	1	2	3	4	3	4	5	6	7	8	9	9	9	9
8	1	2	3	4	3	4	5	6	7	8	9	10	10	10
9	1	2	3	4	3	4	5	6	7	8	9	10	11	11
10	1	2	3	4	3	4	5	6	7	8	9	10	11	12

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