International Journal of Applied Science and Mathematics
Volume 4, Issue 6, ISSN (Online): 2394-2894

Operator Equations, Operator Inequalities and Power Bounded Operators in Hilbert Spaces

Ms Nyamusi Dorca Stephen* and Dr Mutie Kavila
*Corresponding author email id: dostphn@gmail.com

Date of publication (dd/mm/yyyy): 17/11/2017

Abstract **– This is a study on some operator equations, operator inequalities and power bounded operators in Hilbert spaces. Looking at the operator equation TW = WS various properties on T, W and S such as; quasinormal, posinormal, hyponormal among others are satisfied, also on some operator inequalities the equivalence of constability of sequences of norms and its decomposition among other results are shown.**

Keywords **– Hilbert Space, Operator Equation, Operator Inequality and Power Bounded Operators.**

I. Introduction

Prior to the development of Hilbert spaces, there were other generalizations of the Euclidean space which were well known by mathematicians and physicists, for instance; an abstract linear space studied towards the end of the 19th century.

During the first decade of the 20th century, parallel developments led to the introduction of Hilbert spaces. The first of these was the observation which arose during David Hilbert and Erhard Schmidt's study of integral equations which illustrates how two square integrable real-valued functions f and g on an interval [a, b] have an inner product which has many of the familiar properties of the Euclidean dot product by Heine Lebesgue 1904. This was given by;

$\langle f, g\rangle = \int_a^b f(x)g(x)dx$

In particular, the idea of an orthogonal family of functions gained meaning here.

Schmidt exploited the similarity of this inner product with the usual dot product to prove an analog of the spectral decomposition for an operator of the form;

$f(x) \to \int_a^b k(x,y)f(y)dy$

where k is a continuous function symmetric in x and y. The resulting eigen function expansion expresses the function k as a series of the form;

$k(x,y) = \sum_n \lambda_n \varphi_n(x)\varphi_n(y)$

where λ, x, y are eigen functions for $n, m \in \mathbb{N}$ and the functions φ_n are orthogonal in the sense that;

$\langle \varphi_n, \varphi_m\rangle = 0 \; for\ all\ n \neq m$ where $n, m \in \mathbb{N}$.

II. Importance of Hilbert Spaces and Applications

Hilbert spaces support the generalizations of simple geometric concepts like projection and change of basis from their usual finite dimensional setting. Particularly, the special theory of continuous self-adjoint linear operators on a Hilbert space generalizes the usual spectral decomposition of a matrix, and this often plays a major role in applications of the theory to other areas of Mathematics and Physics.

III. Literature Review

Goya and Saito (1981) made a contribution on bounded linear operators on the Hilbert space H denoted as B(H). Goya generalized the Putnam-Fuglede Theorem by showing that if $T, S\ and\ W \in B(H)$ where W has a dense range, then assuming $TW = WS$ and $T^*W = WS^*$ then T and S satisfy the conditions hyponormal, coisometry and normal.

Furuta (1982) studied about the Hilbert Schmidt operators associated with the Putnam-Fuglede Theorem where he proved that if A and B^* were Hyponormal operators and C a Hyponormal operator commuting with A^* and D^* being a Hyponormal operator commuting with B, then for an Hilbert Schmidt operator X, the Hilbert Schmidt norm of AXD + CXB is greater than or equal to the Hilbert Schmidt norm of $A^*XD + C^*XB^*$. In particular;

AXD = CXB which implies that; $A^*XD = C^*XB^*$

IV. Operator Equations in Hilbert Spaces

Remark 1.1

Let the polar decomposition of W^* is given by; $W^* = V^*B$ where B is a positive operator and V^* a coisometry.

Lemma 1.2

Let T, S and W∈ B(H) where W has a dense range. If, $TW = WS$. Then; $T^*W = WS^*$

Proof

Let $W^* = V^*$B be the polar decomposition of W*, B be a non-negative operator and V* be a coisometry with $W^* \in B(H)$. Since W is injective it never maps distinct element of its domain to the same elements of its co-domain. Thus W* is injective. Taking the adjoint on both sides of equation;

$W^* = V^*$B

We then have;

$(W^*)^* = (V^*B)^*$

Which implies;

$W = B^*V$

Thus the equation,

$TW = WS$

Becomes;

$TB^*V = B^*VS$

By post multiplying both sides of, $TB^*V = B^*VS$ with W^* we have,

$TB^*VV^*B = B^*VSV^*B$... (i)

Since V is a coisometry, we then have

$TB^*VV^*B = TB^*IB = TB^*B$ .... (ii)

Now,

$WW^* = B^2$

If and only if B is self-adjoint. For we have;

$WW^* = (B^*V)(V^*B) = B^*VV^*B = B^*B = BB = B^2$

Therefore, $B^2 = WW^*$ is injective and V is coisometric since $VV^* = I$.

Then from the equation (ii) we have;

$TB^*B = TB^2 = TWW^*$ ....(iii)

Combining equations (i) and (iii) we have;

$TWW^* = B^*VSV^*B$

This implies;

$TWW^* = WSW^*$ ....(iv)

By taking the adjoint on both sides of equation (iv) we have;

$(TWW^*)^* = (WSW^*)^*$

This implies,

$WW^*T^* = WS^*W^*$

Thus this implies,

$W^*T^* = S^*W^*$ ....(v)

Letting the operators T and S to be self-adjoint operators, then equation (v) becomes;

$W^*T = SW^*$ ....(vi)

Taking the adjoint on both sides, equation (vi) becomes;

$(W^*T)^* = (SW^*)^*$

This implies;

$T^*W = WS^*$

Lemma 1.3

Let T, S and W∈ B(H) where W has a dense range. If $TW = WS$ and $T^*W = WS^*$, then, $V^*VS = SV^*V$

Where V satisfies $W^* = V^*B$ with B a non-negative operator and V^* is a coisometry with $W^* \in B(H)$

Proof

Since W has a dense range, it never maps distinct elements of its domain to the same element of its co-domain thus W^* is injective. Taking the adjoint on both sides of $W^* = V^*B$ we have;

$(W^*)^* = (V^*B)^*$

This implies;

$W = B^*V$

By post multiplying both sides by W^* we have;

$WW^* = (B^*V)(V^*B) = B^*VV^*B$

Since V is a coisometry $VV^* = I$

Thus;

$B^*VV^*B = B^*IB = B^*B$

Now, $WW^* = B^2$ if and only if B is a self-adjoint operator. Thus;

$WW^* = B^*B = BB = B^2$

Therefore, $B^2 = WW^*$ is injective and V is coisometric since $VV^* = I$.

From the equation $TW = WS$, by post multiplying both sides by W^* we have;

$TWW^* = WSW^*$ ....(i)

Taking the adjoint on both sides of (i) we have; $(T^*WW^*)^* = (WS^*W^*)^*$

This implies;

$WW^*T = WSW^*$ ....(ii)

From the equations (i) and (ii) we have;

$TWW^* = WSW^*$ and $WW^*T = WSW^*$

Thus WW^* commutes with T. Since V is a coisometry, by pre multiplying the operator T with B and post multiplying it with V, where B commutes with T we have;

$BTV = TBV$ ....(iii)

Since B commutes with T, we have;

$TBV = TB^*V$

Since B is self adjoint. Therefore,

$BTV = TB^*V = TW = WS = B^*VS = BVS$ .... (iv)

This implies; $TV = VS$

Since B is injective and V is coisometric, then we have;

$T = TVV^* = VSV^*$ ………..…………… (*)

Since from equation (iii) we have;

$BTV = TBV$

Then this equation implies that;

$BT = TB$

This also implies that;

$TB = BT$ .... (v)

Similarly, Equation (ii) can be written as;

$W^*T = SW^*$ .... (vi)

Thus from the equations (v) and (vi), by pre multiplying (v) with V^* we have,

$V^*TB = V^*BT = W^*T = SW^* = SV^*B$

This implies;

$V^*T = SV^*$

Hence by pre multiplying $VS = TV$ by V^* both sides we have; $V^*VS = V^*TV = SV^*V$

Therefore;

$V^*VS = SV^*V$

Lemma 1.4

Let T, S and $W \in B(H)$, where W has a dense range. Given that;

$TW = WS$ and

$T^*W = WS^*$,

then; $TV = VS$.

Proof

Let $W^* = V^*B$ be the polar decomposition of W^* where B is a non-negative operator and V^*a coisometry with $W^* \in B(H)$. Since W has a dense range, it never maps distinct elements of its domain to the same element of its codomain and thus $W^*$is injective.

Taking the adjoint on both sides of $W^* = V^*B$, we have;

$(W^*)^* = (V^*B)^*$

This implies that;

$W = B^*V$

By post multiplying by W^* we have;

$WW^* = (B^*V)(V^*B) = B^*VV^*B = B^*IB = B^*B$

Now, $WW^* = B^2$

If and only if B is self-adjoint. Thus;

$WW^* = B^*B = BB = B^2$

Therefore, $B^2 = WW^*$ is injective and V is coisometric since $VV^* = I$.

From the equation $TW = WS$, by postmultiplying both sides by W^*we have;

$TWW^* = WSW^*$ .... (i)

And from the equation $T^*W = WS^*$, by postmultiplying both side by W^*, we have; $T^*WW^* = WS^*W^*$

Taking the adjoint on both sides we have; $(T^*WW^*)^* = (WS^*W^*)^*$

This implies;

$WW^*T = WSW^*$ .... (ii)

This also implies that;

$W^*T = SW^*$ .... (iii)

From the equations (i) and (ii), we have; $TWW^* = WSW^*$ and $WW^*T = WSW^*$

Thus WW^* commutes with T and so B^2 commutes with T since $B^2 = WW^*$

Since V is a coisometry, by multiplying the operator T with B where B commutes with T and post multiplying the operator T with V, we have;

$BTV = TBV$

Since B is self-adjoint, we then have;

$TBV = TB^*V = TW = WS = B^*VS = BVS$ ....(iv)

This then implies that;

$TV = VS$ ....(v)

Which also implies $BT = TB$ hence,

$TB = BT$ ....(vi)

Since B is injective and V is coisometric, then we have;

$T = TVV^* = VSV^*$

Hence;

$T = VSV^*$ ....(vii)

By pre multiplying both sides of (vi) by V^* we have; $V^*TB = V^*BT = W^*T = SW^* = SV^*B$,

This implies;

$V^*T = SV^*$ .... (viii)

Also by pre multiplying (v) by V^*, we have ; $V^*VS = V^*TV = SVV^*$

Since V is isometric, this implies;

$S = V^*TV$ ....(ix)

Using (viii) and (ix) we have; $VSV^*V = VV^*TV$

This implies;

$VV^*VS = VV^*TV$ and therefore our proof; $TV = VS$

Corollary 1.5

Let T, S and $W \in B(H)$, where W has a dense range. Given that $TW = WS$ and $T^*W = WS^*$, then;

i. If S is Quasinormal, then T is Quasinormal.
ii. If S is Paramormal, then T is Paranormal.
iii. If S is P-hyponormal, then T is P-hyponormal.
iv. If S is Semi-hypormal, then T is Semi-hyponormal.
v. If S is log-hyponormal, then T is log-hyponormal.

Proof

To prove (i)

Given $S(S^*S) = (S^*S)S$. Then since $T = VSV^*$ from lemma 2.4 we have;

$T(T^*T) = VSV^*[(VSV^*)^*(VSV^*)] = VSV^*VS^*V^*VSV^* = VV^*VSS^*SV^*VV^* = IVSS^*SV^*I = VSS^*SV^* = VS^*SSV = VV^*VS^*SSVV^* = [(VSV^*)^*(VSV^*)]VSV^* = (T^*T)T$

Therefore, we have; $T(T^*T) = (T^*T)T$

To prove (ii)

Since $\| S^2x \| \geq \| Sx \|^2$, then we have; $\| T^2x \| = \| (VSV^*)^2x \| = \| (VSV^*)(VSV^*)x \| = \| VSSV^*x \| \geq \| VSV^*x \|^2$

Since $T = VSV^*$ then;

$\| VSV^*x \|^2 = \| Tx \|^2$

Therefore, $\| T^2x \| \geq \| Tx \|^2$

To prove (iii)

Since $(S^*S)^* \geq (SS^*)^P$, then we have;

$(T^*T)^p = [(VSV^*)^*(VSV^*)]^p = [(VS^*V^*VSV^*)^p] = [(VV^*VS^*SV^*)^p] \geq [(VSS^*V^*)^p] = [(VSV^*)(VSV^*)^*]^p = (TT^*)^p$.

Thus;

$(T^*T)^p \geq (TT^*)^p$

To prove (iv)

Since $(S^*S)^{\frac{1}{2}} \geq (SS^*)^{\frac{1}{2}}$,

Then we have;

$(T^*T)^{\frac{1}{2}} = [(VSV^*)^*(VSV^*)]^{\frac{1}{2}} = [VS^*V^*VSV^*]^{\frac{1}{2}} \geq [VSS^*V^*]^{\frac{1}{2}} \geq [(VSV^*)(VSV^*)^*]^{\frac{1}{2}} = (TT^*)^{\frac{1}{2}}$

Thus, $(T^*T)^{\frac{1}{2}} \geq (TT^*)^{\frac{1}{2}}$

To prove (v)

Since $\log(S^*S) \geq \log(SS^*)$, then we have;

$\log(T^*T) = \log[(VSV^*)^*(VSV^*)] \geq \log[VSS^*V^*] = \log[(VSV^*)(VSV^*)^*] = \log[TT^*]$

Therefore,

$\log(T^*T) \geq \log(TT^*)$

Corollary 1.6

Let T, V and $W \in B(H)$ where T is a paranormal, V is a coisometry, and W has a dense range. If $TW = WV^{*n}$. Then if T is normal with W injective and has a dense range then V is normal.

Proof

Let $x \in H$ be a unit norm such that $Wx \neq 0$ and define $y_n = WV^{*n}x$ (n=0, 1, 2, …). Then by using the Theorem above,

$Ty_{n+1} = TWV^{*n+1}x = WVV^{*n+1}x = WVV^*V^{*n}x = WV^{*n}x = y_n$ … ….(*)

Thus, $Ty_{n+1} = y_n$

By introducing norms to (*) above we have;

$\| y_n \| = \| Ty_{n+1} \| = \| TWV^{*n+1} \|$

Since by hypothesis $TW = WV$, then;

$\| TWV^{*n+1}x \| = \| WVV^{*n+1}x \| = \| WVV^*V^{*n}x \| = \| WV^{*n}x \|$

(Since V is coisometric).

Since W is injective and has a dense range, then V is normal. This implies

$\| Vx \| = \| V^*x \| \quad \forall x \in H$

And thus we have;

$\langle Vx, Vx \rangle = \langle V^*x, V^*x \rangle$

This implies; $\langle x, V^*Vx \rangle = \langle x, VV^*x \rangle$

Thus we have; $V^*V = VV^*$

V. Conclusion

From section IV some important results in the area were proved such as; Goya and Saito (1981) generalized the Putnam-Fuglede Theorem by showing that if $T, S \text{ and } W \in B(H)$ where W has a dense range, then assuming $TW = WS$ and $T^*W = WS^*$ then T and S satisfy the conditions hyponormal, coisometry and normal.

VI. Notations

1. PWB(H): power bounded operator in a Hilbert Space H.
2. R(T): The range of an operator T.

3. H: Hilbert space for the complex scalars C.
4. $\|x\|$: norm of a vector x.
5. $\|T\|$: The operator norm of T.
6. $\langle x, y\rangle$: The inner product of x and y.
7. $x \oplus y$: The direct sum of x and y.
8. $|T|$: The absolute value $(T^{*}T)^{\frac{1}{2}}$ of an operator T.
9. $\in$: Member of
10. $\forall$: for all
11. $B(H)$: Bounded operator in a Hilbert space H.

References

[1] Furuta, T. (1982). A Hilbert-Schmidt Norm Inequality Associated with the Putnam-Fuglede Theorem. *Bull. Austral. Math. Soc.*, 25, 177-185.

[2] Goya, E. and Saito, T. (1981). On Intertwining by an Operator Having a Dense Range. *Toh. Math. Journ.*, (33), 127-13.

Outside Links

[3] https://en.wikipedia.org/wiki/Hilbert_space.

[4] https://en.wikipedia.org/wiki/Essential_Spectrum.

Authors' Profiles

Nyamusi Dorca Stephen Joined Kenyatta University and persued Bachelor of Education Science from 2009 to 2013. She graduated with Second Class Honors (Upper Division). She later Joined Kenyatta University for Msc in Pure Mathematics from 2014 to 2015 and graduated on December 2015 with a project on operator equations, operator inequalities and power bounded operators in Hilbert Spaces. Currently, she is a Tutorial fellow in Technical University of Mombasa.

Dr Mutie Kavila is a Lecturer at Kenyatta University, He received his Msc (Pure Mathematics) in 2005 and Phd (Pure Mathematics) in 2013 from the University of Nairobi. His duties include; Research, supervision and Lecturing of post-graduate and undergraduate students.
email id: mutiekavila@gmail.com